Introduction to Mathematical Micromagnetics

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Lecture 1. Basic micromagnetics

- Magnetic materials display magnetization (density of magnetic dipole moments) in response to magnetic field;
- In ferromagnets spontaneous magnetization occurs at finite temperature in the absence of applied field (e.g. iron, cobalt, nickel and alloys);
- The interaction between neighboring atoms' magnetic dipoles lead to formation of regions with "ordered states";
- It is possible to introduce an order parameter $\mathbf{M} \in M_s\mathbb{S}^2$ – magnetization – describing magnetic behavior of a ferromagnet;
- Observable (metastable) states in ferromagnets can be linked to local minimizers of a miromagnetic energy (depending on $\mathbf{M}$).

Micromagnetic variational principle

The micromagnetic variational principle explains a metastable magnetization distribution in a ferromagnetic sample. The micromagnetic energy has the form (see Landau, Lifshitz (1935); Hubert, Schäfer (1998))

$$E(\mathbf{M}) = \frac{A}{M_s^2}\int_\Omega |\nabla \mathbf{M}|^2 + K\int_\Omega \phi(\mathbf{M}/M_s) + \mu_0 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla\cdot\mathbf{M}(\mathbf{r})\,\nabla\cdot\mathbf{M}(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|} - \mu_0\int_\Omega \mathbf{M}\cdot\mathbf{H}, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^3$, $\mathbf{M} = (M_1, M_2, M_3) = (\overline{\mathbf{M}}, M_3)$, with

$$|\mathbf{M}(\mathbf{r})| = \begin{cases} M_s & \text{when } \mathbf{r} \in \Omega \\ 0 & \text{when } \mathbf{r} \in \mathbb{R}^3 \setminus \Omega \end{cases} \quad (1.2)$$

- M_s is saturation magnetization $\sim 10^6$ A/m;
- A is the exchange stiffness $\sim 10^{-11}$ J/m;
- K is the magnetic anisotropy constant $\sim 0 - 10^7 J/m^3$;
- $\mu_0 = 4\pi \cdot 10^{-7}$ H/m is the permeability of vacuum;
- $\mathbf{H} = \mathbf{H}(x, y, z)$ is the applied magnetic field.

We introduce non-dimensional magnetization vector $\mathbf{m}$ as

$$\mathbf{m} = (m_1, m_2, m_3) \equiv \frac{\mathbf{M}}{M_s}, \quad |\mathbf{m}| = 1 \text{ in } \Omega, \text{ and } \mathbf{m} = 0 \text{ in } \mathbb{R}^2 \setminus \Omega. \tag{1.3}$$

We also introduce an *exchange length* parameter ℓ and anisotropy quality factor Q

$$\ell = \sqrt{\frac{2A}{\mu_0 M_s^2}}, \quad Q = \frac{2K}{\mu_0 M_s^2} \tag{1.4}$$

Units of ℓ are meters ($[\ell] = \mathrm{m}$) and Q is non-dimensional. We rescale domain by ℓ and energy by $A\ell$ to obtain the rescaled energy (Ω is non-dimensional)

$$E(\mathbf{m}) = \int_\Omega |\nabla \mathbf{m}|^2 \, d^3r + Q \int_\Omega \Phi(\mathbf{m}) d^3r \tag{1.5}$$
$$+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{m}(\mathbf{r}) \, \nabla \cdot \mathbf{m}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, d^3r \, d^3r' - 2 \int_\Omega \mathbf{m} \cdot \mathbf{h} \, d^3r,$$

where $\mathbf{h} = \mathbf{H}/M_s$.

In many cases we also need to rescale non-dimensional Ω to obtain a domain of "right size" and it introduces a parameter in front of exchange energy.

Difficulties with full micromagnetic energy

The study of full 3D micromagnetic energy poses a significant challenge to rigorous analysis. It leads to a variational problem with the following properties

(i) multidimensional (domain is 3D);

(ii) vectorial (taking values in $\mathbb{R}^3$);

(iii) nonlinear (constraint $|\mathbf{M}| = M_s$);

(iv) nonlocal (magnetostatic energy);

(v) multiscale (geometry and magnetic parameters).

- **We typically want to study reduced models depending on specific regimes.**

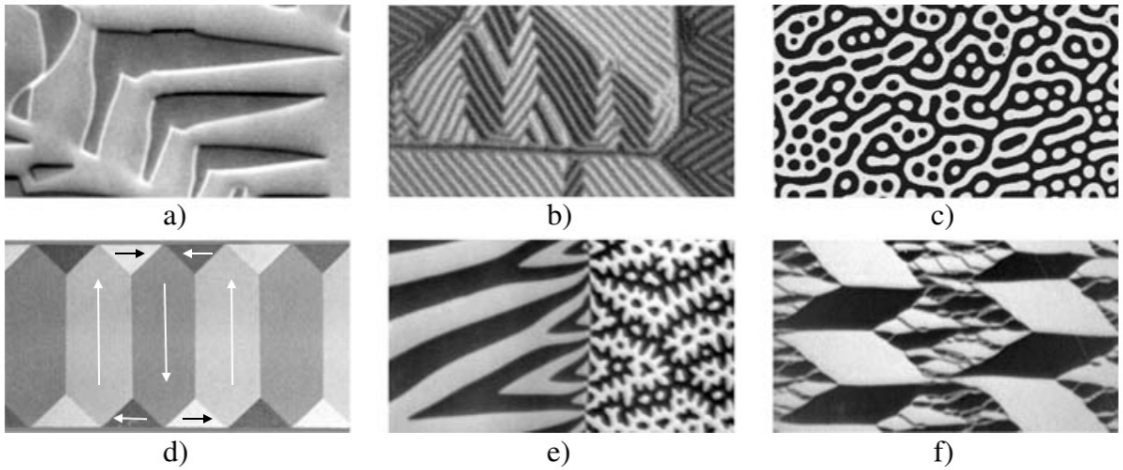

Figure: Magnetic patterns (Hubert, Schäfer (1998))

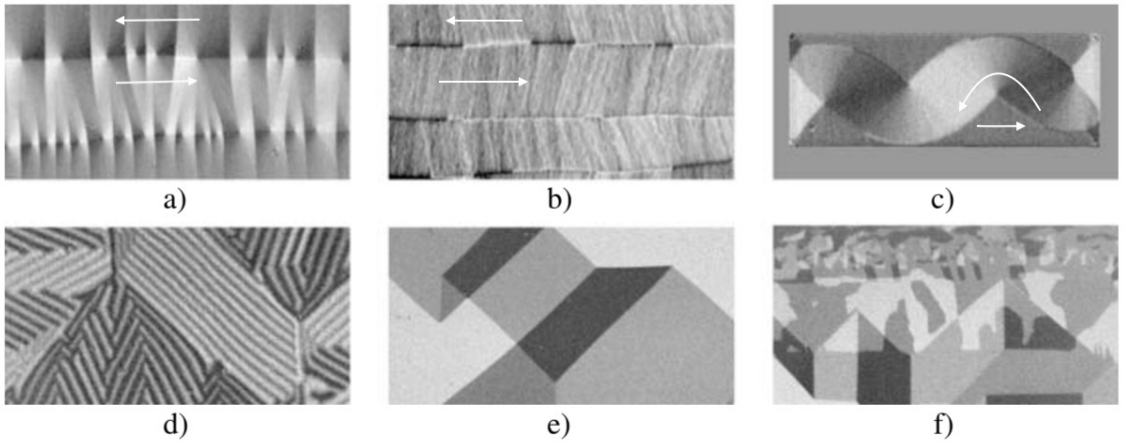

Figure: Magnetic patterns (Hubert, Schäfer (1998))

Basic regimes

- Small particles – "0"-dimensional magnets.

  Magnetization is constant, nonlocal term becomes a quadratic form defined using a constant tensor.
- Thin wires – "1"-dimensional objects.

  Magnetization depends on one variable, nonlocal term simplifies.
- Thin films – "2-dimensional objects. Magnetization depends on two plane variables and there are many different regimes leading to
  - interior and boundary vortices
  - Neel and edge domain walls
  - charged domain walls
  - skyrmions, etc

Mathematical questions we want to address

- Derivation of reduced models
- Existence (or conditions for existence) of certain magnetic patterns (domain walls, vortices, skyrmions, etc)
- Uniqueness of magnetic patterns in specific material parameter regimes
- Internal structure (or profile) of magnetic patterns
    - symmetry and symmetry breaking phenomena
    - monotonicity of a profile
    - behavior at infinity (how fast profile decays)
    - boundary layers, etc
- Minimal energy scalings

Basic structures: 1D domain walls

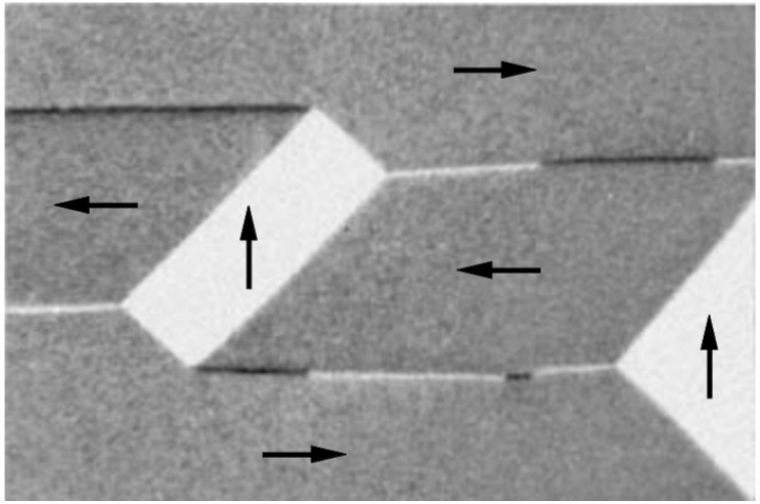

Figure: Magnetic Domains

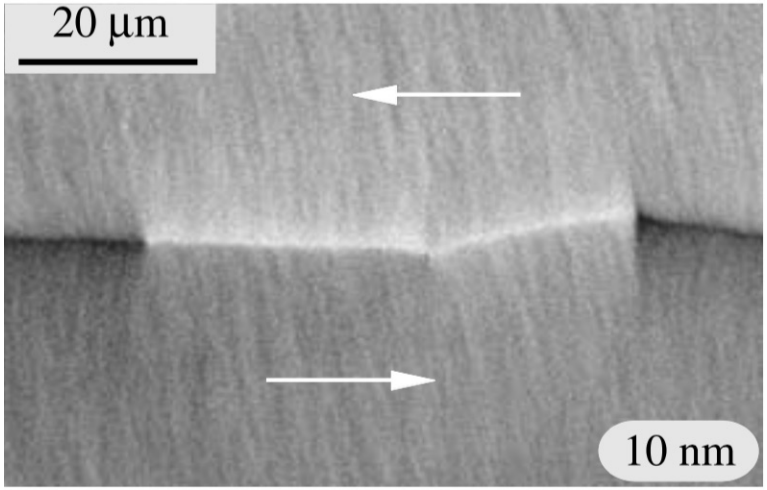

Figure: Néel wall

1D domain walls - simplified energy

Let us look at 1D example of domain walls. We define

$$\mathcal{E}(\mathbf{m}) = \varepsilon^2 \int_{\mathbb{R}} |\mathbf{m}'|^2 \, dx + Q \int_{\mathbb{R}} m_1^2 + m_2^2 \, dx \tag{1.6}$$

After rescaling $x \to \frac{Q^{1/2}}{\varepsilon} x$ we obtain $\mathcal{E}(\mathbf{m}) = Q^{\frac{1}{2}} \varepsilon \; E(\mathbf{m})$, where

$$E(\mathbf{m}) = \int_{\mathbb{R}} |\mathbf{m}'|^2 \, dx + \int_{\mathbb{R}} m_1^2 + m_2^2 \, dx. \tag{1.7}$$

We want to find local minimizers of $E(\mathbf{m})$ in the class

$$\mathcal{A} = \{\mathbf{m} \in H^1_{loc}(\mathbb{R}; \mathbb{R}^3) : |\mathbf{m}(x)| = 1, \mathbf{m}', \; m_1, \; m_2 \in L^2(\mathbb{R}; \mathbb{R}^3)\}. \tag{1.8}$$

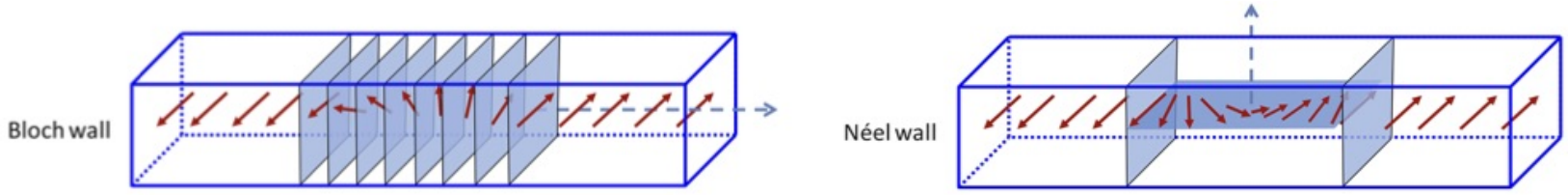

Figure: (a) Bloch and (b) Néel walls (Bhattia et. al. (2019)

1D domain walls - local minimality

Step 1. In order to do it we minimize $E(\mathbf{m})$ in the class

$$\mathcal{A}_1 = \{\mathbf{m} \in H^1_{loc}(\mathbb{R};\mathbb{R}^3) : |\mathbf{m}(x)| = 1, \mathbf{m}' \in L^2(\mathbb{R};\mathbb{R}^3), \mathbf{m}(x) \to \pm\mathbf{e}_3, \text{as } x \to \pm\infty\}$$

using Modica-Mortola trick (Modica, Mortola, *Bol. U.M.I.* (1977))

We reduce problem to one variable and find lower bound on the energy

$$|m_3'|^2 m_3^2 = |\mathbf{m}_\perp \cdot \mathbf{m}_\perp'|^2 \leq |\mathbf{m}_\perp|^2 |\mathbf{m}_\perp'|^2, \quad |\mathbf{m}'|^2 \geq \frac{|m_3'|^2}{1-m_3^2} \tag{1.9}$$

$$E(\mathbf{m}) \geq \int_{\mathbb{R}} \frac{|m_3'|^2}{1-m_3^2}\, dx + \int_{\mathbb{R}} (1-m_3^2)\, dx \tag{1.10}$$

$$= \int_{\mathbb{R}} \left(\frac{|m_3'|}{\sqrt{1-m_3^2}} - \sqrt{1-m_3^2} \right)^2 dx + 2\int_{\mathbb{R}} |m_3'|\, dx \geq 4 \tag{1.11}$$

1D domain walls - local minimality

Step 2. We solve on $\mathbb{R}$

$$m_3' = (1 - m_3^2), \;\; m_3(x) \to \pm 1 \text{ as } x \to \pm\infty \tag{1.12}$$

to obtain $m_3 = \tanh(x)$. The minimizer $\mathbf{m}$ is

$$\mathbf{m} = \operatorname{sech}(x)(\cos\phi_0, \sin\phi_0, 0) + \tanh(x)\mathbf{e}_3, \tag{1.13}$$

where $\phi_0 \in [0, \pi)$.

Step 3. We observe that $\mathbf{m} = \operatorname{sech}(x)(\cos\phi_0, \sin\phi_0, 0) + \tanh(x)\mathbf{e}_3$ satisfies Euler-Lagrange equation for critical points of E in $\mathcal{A}$. Moreover, for $\mathbf{v} \in \mathcal{A}$ such that $|\mathbf{m} + t\mathbf{v}| = 1$ and $|t| \leq \delta$ for $\delta > 0$ small enough we immediately have $E(\mathbf{m} + t\mathbf{v}) - E(\mathbf{m}) \geq 0$ (as boundary conditions are preserved) and therefore $\mathbf{m}$ is a local minimizer with respect to H^1 perturbations.

Basic types of 1D domain walls

Local energy does not distinguish between DWs depending on ϕ_0 but magnetostatic energy does. Major types of 1D DWs are

- Bloch wall is typically observed in bulk ferromagnets ($\nabla \cdot \mathbf{m} = 0$)

$$\mathbf{m} = \operatorname{sech}(x)\mathbf{e}_2 + \tanh(x)\mathbf{e}_3 \tag{1.14}$$

- Neel wall is typically observed in thin ferromagnetic films with in-plane magnetization (profile resolutions requires *magnetostatic energy*)

$$\mathbf{m} = \operatorname{sech}(x)\mathbf{e}_1 + \tanh(x)\mathbf{e}_3 \tag{1.15}$$

- Charged wall ($\mathbf{m}(x) \to \pm\mathbf{e}_1$ as $x \to \pm\infty$) is typically observed in nanowires with easy axis along the wire (profile resolution requires *magnetostatic energy*)

$$\mathbf{m} = \tanh(x)\mathbf{e}_1 + \operatorname{sech}(x)(0, \sin\phi_0, \cos\phi_0) \tag{1.16}$$

Remark. On a finite interval $(-a, a)$ there are no non-trivial local minimizers of E and all critical points are unstable (see Casten, Holland, *J. Diff. Eq.* (1978)). We need non-local part of magnetostatic energy to obtain non-trivial local minimizers.

Néel wall profile

1D Néel wall has been extensively studied (e.g. Melcher, *ARMA* (2003); DeSimone, Knüpfer, Otto, *CVPDE*, (2006); Chermisi, Muratov, *Nonlinearity*, (2013))

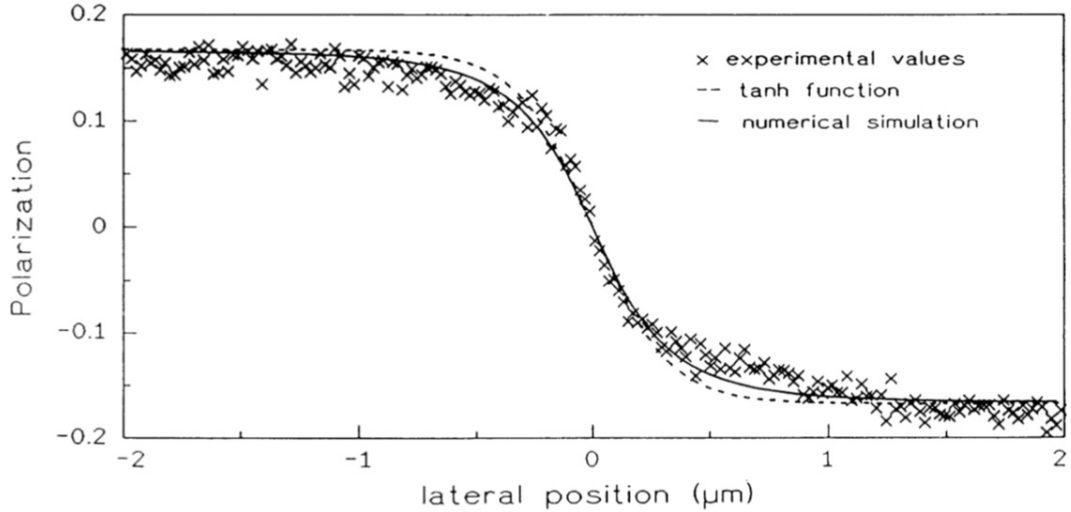

FIG. 1. Line scan across a 180°-domain wall for 5.5 monolayers Co/Cu(100). The experimental data points (×) are compared with a tanh function (dashed line) and the numerical calculation based on the one-dimensional micromagnetic description (solid line; $A = 1.3 \times 10^{-6}$ erg/cm, $K = -2000$ erg/cm^3, $M_s = 750$ emu, $b = 1.0$ nm).

Figure: Structure of Néel wall (Berger, *Phys. Rev. B* (1992))

Charged walls profiles

The structure of charged domain walls is still not completely understood mathematically (Morini, Muratov, Novaga, S, *ARMA* (2023))

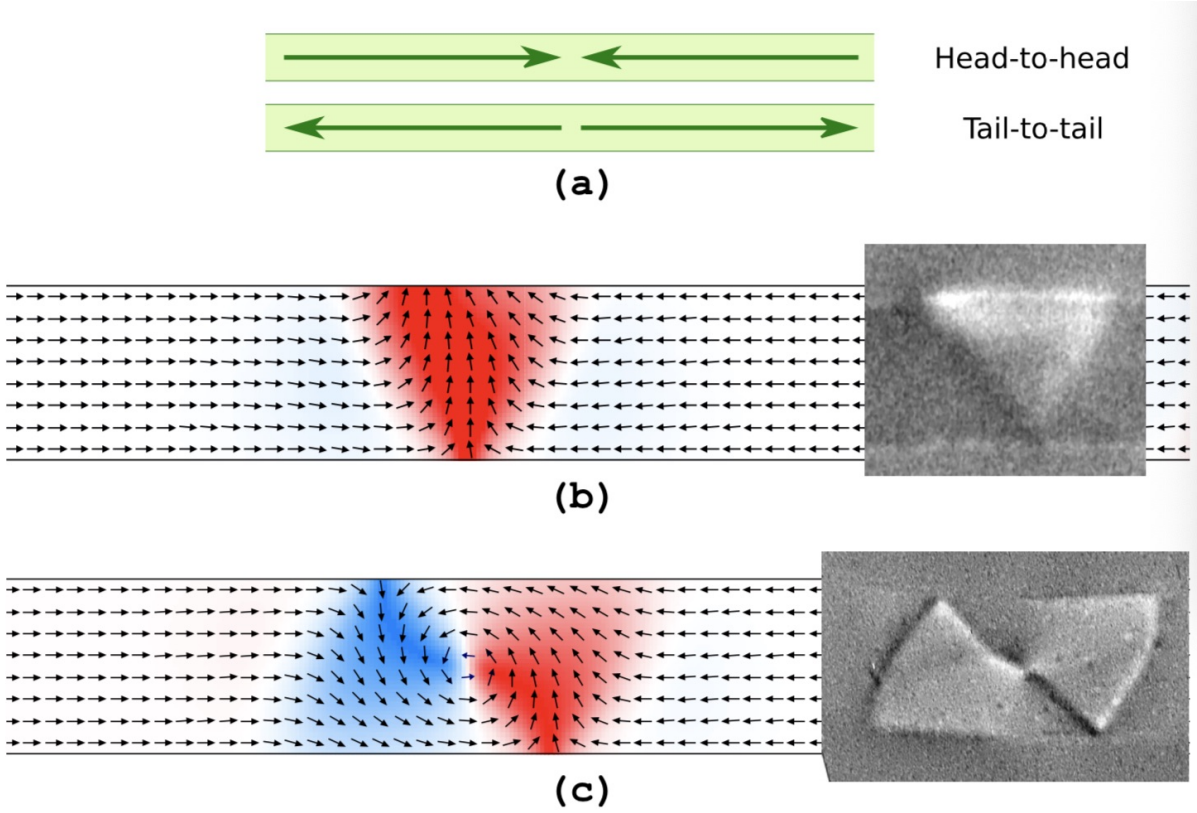

Figure: (a) Schematic charged walls; (b) Structure of transverse CW; (c) Structure of vortex CW (Fruchart (2018))

Basic structures: 2D vortex

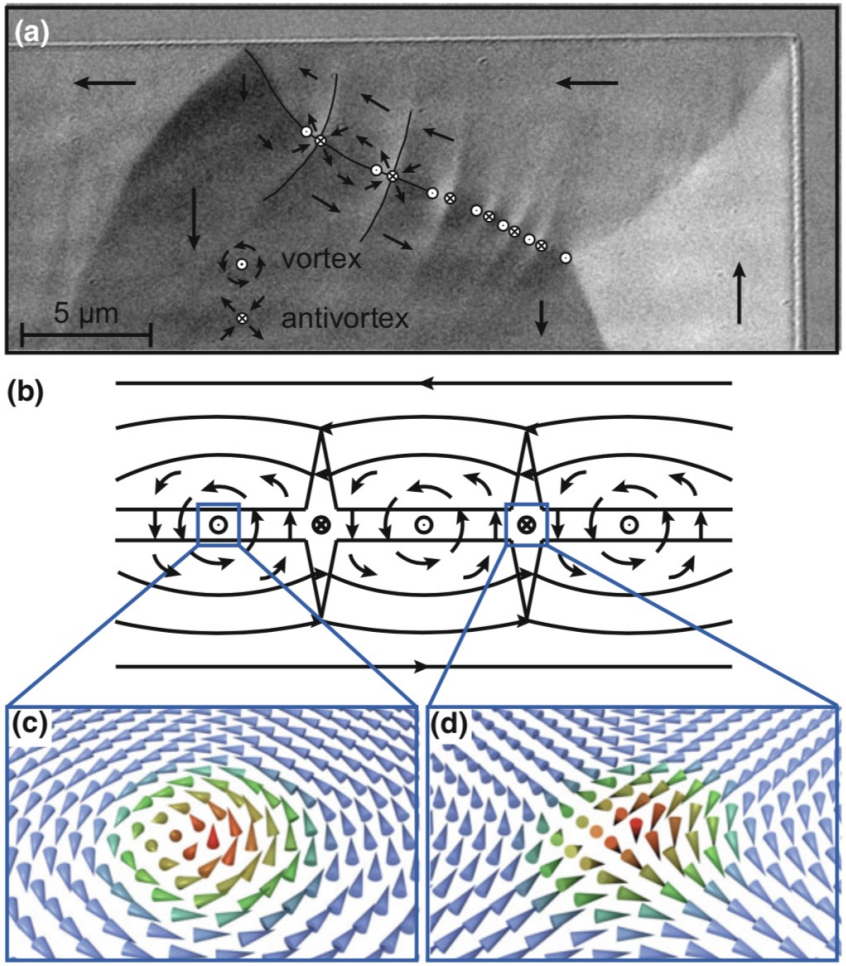

Figure: Vortices and antivortices in cross-tie walls (Behncke et al. (2018)

2D vortex - Ginzburg-Landau model

Let us recall basic Ginzburg-Landau model (*JETP* (1950))

$$E(\mathbf{u}) = \int_B |\nabla \mathbf{u}(\mathbf{r})|^2 + \frac{1}{\varepsilon^2}(1 - |\mathbf{u}|^2)^2 \, d^2 r, \tag{1.17}$$

where $B \subset \mathbb{R}^2$ is a unit ball, $\mathbf{u} : B \to \mathbb{R}^2$, $\mathbf{u}(x) = \tau(x) \equiv (-y, x)$ on ∂B.

- Minimizer exists and is radial for small enough and large enough ε (Bethuel, Brezis, Helein (1994); Pacard, Riviere (2000))
- Radial solution is unique and is a local minimizer for all $\varepsilon > 0$ (Mironescu, *J. Func. An.* (1995))
- For general boundary condition with winding number $d \in \mathbb{Z} \setminus \{0\}$, as $\varepsilon \to 0$ the minimal energy behaves as (Bethuel, Brezis, Helein (1994))

$$E(\mathbf{u}) \sim 2\pi |\ln \varepsilon| \sum_{i=1}^{|d|} d_i^2 + W(a_j, d_j),$$

where a_j are locations of vortices and $d_j = \mathrm{sgn}(d)$ are their degrees

Magnetic 2D vortex

We want to understand a vortex structure observed in thin films. The basic energy to consider is

$$E(\mathbf{m}) = \varepsilon^2 \int_B |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r + \int_B m_3^2(\mathbf{r}) \, d^2 r \tag{1.18}$$

with $\mathbf{m}(x) = \tau(x) \equiv (-y, x, 0)$ on ∂B and $\varepsilon = \frac{\ell}{L}$.

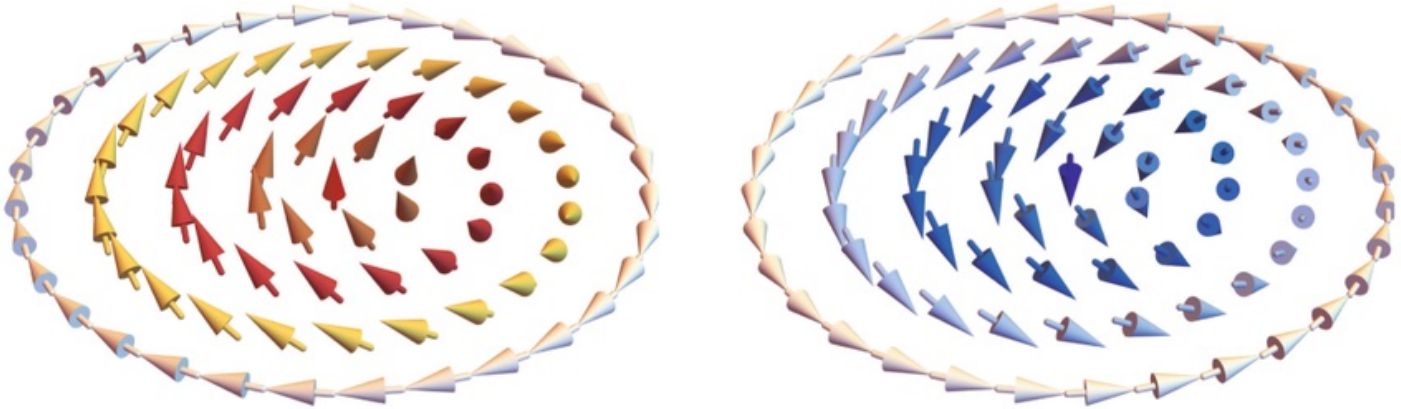

Figure: Magnetic vortices (merons)

Vortex energy bounds

Intuitively, the vortex profile is split in two parts

$$\mathbf{m} \approx \begin{cases} \mathbf{e}_\phi & \text{for } x \in B \setminus B_\varepsilon \\ f(r)\mathbf{e}_\phi + g(r)\mathbf{e}_3 & \text{for } x \in B_\varepsilon \end{cases} \tag{1.19}$$

The energy is concentrated outside of the core B_ε in exchange and scales as

$$E(\mathbf{m}) \sim 2\pi\varepsilon^2 \ln \frac{1}{\varepsilon} \tag{1.20}$$

We can also obtain a lower bound on the energy as

$$E(\mathbf{m}) \geq \varepsilon^2 \int_B |\nabla \mathbf{m}'|^2 \, d^2r + \int_B (1 - |m'|^2)^2 \, d^2r \tag{1.21}$$

$$\geq \int_B \varepsilon^2 |\nabla \rho|^2 + (1-\rho)^2 \, d^2r + \varepsilon^2 \int_B \rho^2 |\nabla u|^2 \, d^2r, \tag{1.22}$$

where $\mathbf{m}' = |\mathbf{m}'| \frac{\mathrm{m}'}{|\mathrm{m}'|} = \rho u$.

Vortex energy bounds

For any $0 < r < 1$ and $u : \partial B_r \to \mathbb{S}^1$ we define winding number

$$d(u; \partial B_r) = \frac{1}{2\pi} \int_{\partial B_r} (u_1 \partial_\tau u_2 - u_2 \partial_\tau u_1)\, d\mathcal{H}^1$$
$$= \frac{1}{2\pi} \int_0^{2\pi} (u_1 \partial_\phi u_2 - u_2 \partial_\phi u_1)\, d\phi \in \mathbb{Z} \quad (1.23)$$

and assume for any $0 < r < 1$ we have $d(u; \partial B_r) = d(u; \partial B) = 1$. Then, defining $\rho_m(r) = \min_{\partial B_r} \rho$, we have

$$E(\mathbf{m}) \geq \int_0^1 \int_{\partial B_r} \varepsilon^2 |\nabla_\tau \rho|^2 + (1-\rho)^2 + \varepsilon^2 \int_0^1 \rho_m^2 \int_{\partial B_r} |\nabla u|^2 \quad (1.24)$$
$$\geq C\varepsilon \int_\varepsilon^1 (1-\rho_m)^2\, dr + 2\pi\varepsilon^2 \int_\varepsilon^1 \rho_m^2 d^2(u; \partial B_r) \frac{1}{r}\, dr \quad (1.25)$$

Now we minimize $\min_{\rho_m \in [0,1]} C\varepsilon(1-\rho_m)^2 + 2\pi\varepsilon^2 \rho_m^2 \frac{1}{r}$ to obtain $\rho_m = \frac{Cr}{Cr + 2\pi\varepsilon}$. Plugging in minimal value in the energy above we obtain

$$E(\mathbf{m}) \geq 2\pi\varepsilon^2 \left(\ln \frac{1}{\varepsilon} - C \right) \quad (1.26)$$

Radial vortex - local minimality

Minimizing energy we obtain harmonic map type equation

$$-\Delta \mathbf{m} + \frac{1}{\varepsilon^2} m_3 \mathbf{e}_3 = \lambda \mathbf{m} = (|\nabla \mathbf{m}|^2 + \frac{1}{\varepsilon^2} m_3^2)\mathbf{m} \tag{1.27}$$

$$\mathbf{m} = (-y, x, 0) \text{ on } \partial B \tag{1.28}$$

We can ask the same questions as for Ginzburg-Landau model.

- We want to investigate **minimality** of axially symmetric solution

$$\mathbf{m} = f(|\mathbf{r}|)\mathbf{e}_\phi + g(|\mathbf{r}|)\mathbf{e}_3, \text{ with } f^2(\mathbf{r}) + g^2(\mathbf{r}) = 1 \text{ and} \tag{1.29}$$

$$-f''(r) - \frac{1}{r} f'(r) + \frac{1}{r^2} f(r) = \lambda(r) f, \quad f(0) = 0, \quad f(1) = 1, \tag{1.30}$$

$$-g''(r) - \frac{1}{r} g'(r) + \frac{1}{\varepsilon^2} g = \lambda(r) g, \quad g(0) = 1, \quad g(1) = 0, \tag{1.31}$$

where $\lambda(r) = |f'(r)|^2 + \frac{1}{r^2}|f(r)|^2 + |g'(r)|^2 + \frac{1}{\varepsilon^2}|g(r)|^2$.

Fact. There exists unique (f, g) solving above equations such that $f > 0, g > 0, f' > 0, g' < 0$ in $(0, 1)$.

Sketch of stability proof

- Compute the second variation

$$\delta^2 E(\mathbf{m})(\mathbf{v}) = \int_B |\nabla \mathbf{v}|^2 \, d^2 r + \frac{1}{\varepsilon^2} \int_B v_3^2 \, d^2 r - \int_B \lambda(r) |\mathbf{v}|^2 \, d^2 r, \tag{1.32}$$

where $\mathbf{v} \in C_c^\infty(B \setminus \{0\}; \mathbb{R}^3)$, $\mathbf{m} \cdot \mathbf{v} = 0$.

- Represent $\mathbf{v}(\mathbf{r}) = u(r, \phi)\mathbf{n}(\phi) + v(r, \phi)\tau(\phi) + w(r, \phi)\mathbf{e}_3$ and use Fourier series representation for u, v and w
- Split $\delta^2 E(\mathbf{m})(\mathbf{v})$ uisng Fourier modes

$$\frac{1}{2\pi} \delta^2 E = F_0(u_0, v_0, w_0) + \frac{1}{2} \sum_{k=1}^{\infty} F_k(u_k, v_k, w_k). \tag{1.33}$$

- Show that for $k \geq 2$ we can control $F_k(u_k, v_k, w_k) \geq F_0(u_0, v_0, w_0)$.
- Use decomposition trick to show $F_0 > 0$ and $F_1 > 0$. For $F_0(u_0, 0, 0)$ we decompose $u_0 = f(r)\xi$, where $\xi \in C_c^\infty(0, 1)$ is arbitrary. In this case using equation for f we have

$$F_0(u_0, 0, 0) = \int_0^1 \left(|f'\xi + f\xi'|^2 + \frac{1}{r^2} |f\xi|^2 - \lambda(r) |f\xi|^2 \right) r dr = \int_0^1 |f\xi'|^2 r dr \geq 0.$$

- Deduce local stability and local minimality (Ignat, Nguyen, *AIHP* (2023)).