Introduction to Mathematical Micromagnetics

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Lecture 2. Basic structures: 2D boundary vortex

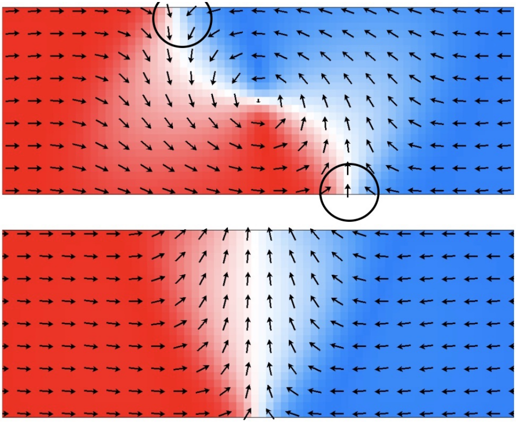

Figure: Vortex and transverse DWs (Jamet et al., (2015))

Basic structures: 2D boundary vortex

We want to understand a structure of a *boundary vortex*. The simplified energy is ($\mathbf{m} \in \mathbb{S}^1$)

$$E(\mathbf{m}) = \varepsilon^2 \int_{\mathbb{R}^2_+} |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r + \int_{\mathbb{R}} m_2^2(0,x)\, dx. \tag{1.1}$$

We cannot talk about minimizers in a usual sense as for vortex boundary conditions

$$\mathbf{m}(\mathbf{r}) \to \frac{(-y,x)}{|\mathbf{r}|} \text{ as } |\mathbf{r}| \to \infty. \tag{1.2}$$

energy is infinite. Instead we want to look at critical points.

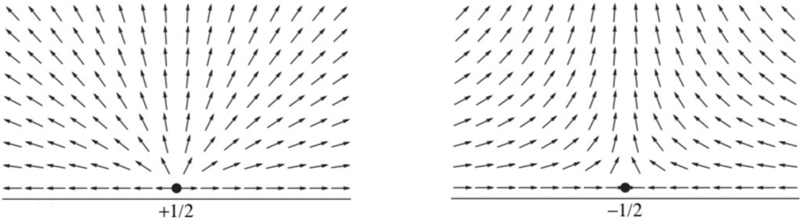

Figure: Boundary vortices (Tchernyshyov, Chern, *PRL*, (2005))

2D boundary vortex - reformulation

We can reformulate the problem as a scalar one and link it to 1D non-local gradient phase transitions. Indeed, using $\mathbf{m} = (\cos\theta, \sin\theta)$ the energy becomes

$$E(\mathbf{m}) \equiv F(\theta) = \varepsilon^2 \int_{\mathbb{R}^2_+} |\nabla\theta|^2 d^2 r + \int_{\mathbb{R}} \sin^2\theta \, d\mathcal{H}^1. \tag{1.3}$$

The critical points/local minimizers of this energy have been extensively studied (e.g. Toland, *J. Func. An.*, (1997); Cabre, Sola-Morales, *CPAM*, (2005); Kurzke, *CVPDE*,(2006); Moser, *ARMA*, (2004))

Let us link the energy (1.3) with some nonlocal 1D energy. We can fix a trace of θ at the boundary to be a given function, $\theta(x,0) = \bar{\theta}(x)$ and then reformulate the problem in terms of $\bar{\theta}$. It amounts to obtain a harmonic extension of $\bar{\theta}$ on a half-plane, or solve

$$\Delta\theta = 0 \text{ in } \mathbb{R}^2_+, \quad \theta(x,0) = \bar{\theta}(x). \tag{1.4}$$

We can then plug this solution into the energy and minimize with respect to $\bar{\theta}$.

2D boundary vortex - nonlocal 1D energy

Let us proceed informally and assume $\bar{\theta} \in C_c^\infty(\mathbb{R})$. We can define the Fourier transform in x variable as

$$\hat{\theta}(k, y) = \int_{\mathbb{R}} e^{-ikx} \theta(x, y)\, dx. \tag{1.5}$$

We also note that $\hat{\theta}(k, 0) = \hat{\bar{\theta}}(k)$. Writing the Euler-Lagrange equations (recall that $\bar{\theta}$ is fixed) we obtain

$$-\partial_y^2 \hat{\theta}(k, y) + k^2 \hat{\theta}(k, y) = 0, \quad \hat{\theta}(k, 0) = \hat{\bar{\theta}}(k), \quad \partial_y \hat{\theta}(k, \infty) = 0. \tag{1.6}$$

The solution of this equation is $\hat{\theta}(k, y) = \hat{\bar{\theta}}(k) e^{-ky}$.
We can now rewrite our energy using Fourier representation as

$$F(\theta) := \varepsilon^2 \int_0^\infty \int_{\mathbb{R}} |\partial_y \hat{\theta}(k, y)|^2 + k^2 |\hat{\theta}(k, y)|^2 \, dk\, dy + \int_{\mathbb{R}} \sin^2 \bar{\theta}(x)\, dx \tag{1.7}$$

2D boundary vortex - nonlocal 1D energy

We plug expression for $\hat{\theta}(k, y)$ into the energy and obtain

$$F(\bar{\theta}) = \varepsilon^2 \int_{\mathbb{R}} \int_0^\infty 2k^2 e^{-2ky} |\hat{\bar{\theta}}(k)|^2 \, dy \, dk + \int_{\mathbb{R}} \sin^2 \bar{\theta}(x) \, dx \tag{1.8}$$

$$= \int_{\mathbb{R}} k |\hat{\bar{\theta}}(k)|^2 \, dk + \int_{\mathbb{R}} \sin^2 \bar{\theta}(x) \, dx. \tag{1.9}$$

Taking inverse Fourier transform we obtain

$$F(\bar{\theta}) = \frac{\varepsilon^2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\bar{\theta}(x) - \bar{\theta}(y)|^2}{|x - y|^2} \, dx \, dy + \int_{\mathbb{R}} \sin^2 \bar{\theta}(x) \, dx. \tag{1.10}$$

We observe that our energy is now informally represented as a non-local gradient phase transition problem

$$F(\bar{\theta}) = \varepsilon^2 \int_{\mathbb{R}} |\partial^{\frac{1}{2}} \bar{\theta}(x)|^2 \, dx + \int_{\mathbb{R}} \sin^2 \bar{\theta}(x) \, dx. \tag{1.11}$$

Various aspects of similar problems have been extensively studied (e.g. Alberti, Bouchitté, Seppecher, *C.R. Acad. Sci. Paris*, (1994); Palatucci, Savin, Valdinoci, *Ann. Mat. Pure Appl.*, (2013))

2D boundary vortex - classification of critical points

The Euler-Lagrange equation is

$$\frac{\varepsilon^2}{\pi}\int_{\mathbb{R}} \left(2\bar{\theta}(x) - \bar{\theta}(x-\xi) - \bar{\theta}(x+\xi)\right)\frac{1}{|\xi|^2}\,d\xi + \sin 2\bar{\theta}(x) = 0 \qquad \forall x \in \mathbb{R}.$$

We can explicitly check that

$$\bar{\theta}(x) = \frac{\pi}{2} \pm \arctan(x/\varepsilon^2) + \pi n \tag{1.12}$$

is a non-trivial solution of this equation.

The Euler-Lagrange equation for original problem is

$$\Delta\theta = 0 \text{ in } \mathbb{R}^2_+, \quad \varepsilon^2\partial_y\theta(x,0) = -\frac{1}{2}\sin 2\theta(x,0). \tag{1.13}$$

We can classify solutions of this problem (Toland, *J. Func. An.*, (1997))

Theorem 1

Let θ be a bounded solution of (1.13). Then either θ is constant solution, periodic solution, or there exists $a \in \mathbb{R}$, $n \in \mathbb{Z}$ such that

$$\theta(x,y) = \frac{\pi}{2} \pm \arctan\frac{x+a}{y+\varepsilon^2} + \pi n. \tag{1.14}$$

Basic structures: 2D baby skyrmion

We want to understand a skyrmion structure in thin films. The simplified energy is

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r \qquad (1.15)$$

defined in the class

$$\mathcal{A} = \{\mathbf{m} \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^3) \; : \; \nabla \mathbf{m} \in L^2(\mathbb{R}^2; \mathbb{R}^3 \times \mathbb{R}^3), \; |\mathbf{m}(x)| = 1 \, a.e.\}. \qquad (1.16)$$

We impose *skyrmion boundary conditions*

$$\mathbf{m}(\mathbf{r}) \to -\mathbf{e}_3 \text{ as } |\mathbf{r}| \to \infty \qquad (1.17)$$

and a topological degree condition

$$\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} (\partial_1 \mathbf{m} \times \partial_2 \mathbf{m}) \cdot \mathbf{m} \, d^2 r = 1. \qquad (1.18)$$

Basic structures: 2D baby skyrmion

It is clear that

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r = \int_{\mathbb{R}^2} |\partial_2 \mathbf{m} - \mathbf{m} \times \partial_1 \mathbf{m}|^2 \, d^2 r + 8\pi \geq 8\pi \qquad (1.19)$$

We want to find a suitable profile such that

$$\partial_2 \mathbf{m} - \mathbf{m} \times \partial_1 \mathbf{m} = 0 \qquad (1.20)$$

This is Belavin-Polyakov profile (Belavin, Polyakov, *JETP Lett.* (1975))

$$\mathbf{m} = \left(\frac{\pm 2x}{1+r^2}, \frac{\pm 2y}{1+r^2}, \frac{1-r^2}{1+r^2} \right). \qquad (1.21)$$

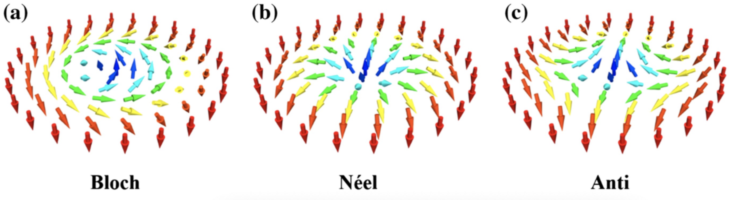

Figure: (a) Bloch skyrmion; (b) Neel skyrmion; (c) Anti-skyrmion

Basic structures: 2D baby skyrmion.

We can formulate a general problem. Consider the energy

$$E(\mathbf{m}) = \int_\Omega |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r - 2\kappa \int_\Omega \nabla m_3 \cdot \mathbf{m} \, d^2 r + (Q-1) \int_\Omega (1 - m_3^2) \, d^2 r + 2h \int_\Omega (1 + m_3) \, d^2 r,$$

where $\mathbf{m} \in \mathcal{A}$, $\Omega \subseteq \mathbb{R}^2$, $\mathbf{m} = -\mathbf{e}_3$ on $\mathbb{R}^2 \setminus \Omega$, $\mathcal{N}(\mathbf{m}) = d \in \mathbb{Z}$.

- Existence of minimizers for $d = 1$ (Melcher, *Proc. R. Soc. Lon. A* (2014); Monteil, Muratov, Simone, S, *Com. Math. Phys.* (2023))
- Existence of minimizers for $d \neq \{0, 1\}$? **partial results**
- What are minimizing profiles? (Li, Melcher, *J. Func. An.* (2018))
- Existence/structure of skyrmion lattice? (Hill, S, Tchernyshev, *SciPost Phys.* (2021))

Lecture 2. Magnetostatic energy.

The magnetostatic / stray field energy is defined as

$$E_{ms}(\mathbf{M}) = -\frac{\mu_0}{2} \int_\Omega \mathbf{H}_d \cdot \mathbf{M} \, d^3 r. \tag{1.22}$$

Here $\mathbf{H}_d$ is a demagnetizing field solving

$$\nabla \cdot (\mathbf{H}_d + \mathbf{M}) = 0, \quad \nabla \times \mathbf{H}_d = 0 \tag{1.23}$$

Introduce a scalar potential $U_d : \mathbb{R}^3 \to \mathbb{R}$ with $\mathbf{H}_d = -\nabla U_d$ and U_d solves

$$\int_{\mathbb{R}^3} \nabla U_d \cdot \nabla \phi \, d^3 r = \int_\Omega \mathbf{M} \cdot \nabla \phi \, d^3 r, \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^3). \tag{1.24}$$

The stray field energy can be rewritten as

$$E_{ms}(\mathbf{M}) = \frac{\mu_0}{2} \int_\Omega \nabla U_d \cdot \mathbf{M} \, d^3 r = \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla U_d|^2 \, d^3 r. \tag{1.25}$$

Solving Poisson equation we obtain

$$E_{ms}(\mathbf{M}) = \frac{\mu_0}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{M}(\mathbf{r}) \, \nabla \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r \, d^3 r'. \tag{1.26}$$

Magnetostatic energy: Min/Max problems.

Rescaling $\mathbf{M}$ to $\mathbf{m}$, U_d to u_m and E_{ms} we can represent

$$E_{ms}(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_m|^2 \, d^3r, \quad \Delta u_m = \nabla \cdot \mathbf{m}. \tag{1.27}$$

Theorem 2

For any $\mathbf{m} \in L^2(\Omega;\mathbb{R}^3)$ with $\Omega \subset \mathbb{R}^3$ bounded there is unique solution $u_m \in H^1(\mathbb{R}^3)$ of the following maximization problem

$$\max_{u\in H^1(\mathbb{R}^3)} \int_\Omega \mathbf{m}\cdot\nabla u \, d^3r - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d^3r. \tag{1.28}$$

Moreover, u_m satisfies Euler-Lagrange equations

$$\int_{\mathbb{R}^3} \nabla u_m \cdot \nabla\phi \, d^3r = \int_\Omega \mathbf{m}\cdot\nabla\phi \, d^3r, \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^3) \tag{1.29}$$

and

$$E_{ms}(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_m|^2 \, d^3r = \max_{u\in H^1(\mathbb{R}^3)} \int_\Omega \mathbf{m}\cdot\nabla u \, d^3r - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d^3r.$$

Magnetostatic energy: vector potential.

There is a way to represent magnetostatic energy through minimizatioin problem. We have Maxwell's equations

$$\nabla \cdot (\mathbf{h}_d + \mathbf{m}) = 0, \quad \nabla \times \mathbf{h} = 0. \tag{1.30}$$

Before we used $\mathbf{h}_d = -\nabla u$ to obtain maximization problem using potential u. Now we can use $\mathbf{h}_d + \mathbf{m} = \nabla \times \mathbf{a}$ with Coulomb gauge $\nabla \cdot \mathbf{a} = 0$. This leads to

$$\nabla \times (\nabla \times \mathbf{a}_m) = -\Delta \mathbf{a}_m = \nabla \times \mathbf{m}. \tag{1.31}$$

The magnetostatic energy is

$$E_{ms} = \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{h}_d|^2 \, d^3 r = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \mathbf{a}_m - \mathbf{m}|^2 \, d^3 r. \tag{1.32}$$

Magnetostatic energy: Min/Min problem.

We can formulate a minimization problem (Di Fratta et. al, *SIMA*, (2020))

Theorem 3

For any $\mathbf{m} \in L^2(\Omega; \mathbb{R}^3)$ *with* $\Omega \subset \mathbb{R}^3$ *bounded there is unique solution* $\mathbf{a}_m \in \mathring{H}^1(\mathbb{R}^3)$ *of the following minimization problem*

$$\min_{\mathrm{a} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \mathbf{a}|^2 + \frac{1}{2}\int_{\Omega} |\mathbf{m}|^2 - \int_{\Omega} \mathbf{m} \cdot \nabla \times \mathbf{a}. \tag{1.33}$$

Moreover, $\mathbf{a}_m$ *satisfies Euler-Lagrange equations*

$$-\Delta \mathbf{a}_m = \nabla \times \mathbf{m} \quad \text{in } \mathring{H}^{-1}(\mathbb{R}^3;\mathbb{R}^3) \tag{1.34}$$

and

$$E_{ms}(\mathbf{m}) = \min_{\mathrm{a} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \mathbf{a}|^2 + \frac{1}{2}\int_{\Omega} |\mathbf{m}|^2 - \int_{\Omega} \mathbf{m} \cdot \nabla \times \mathbf{a}.$$

Useful for localized upper bounds and as a double minimization

$$\min_{\mathrm{m}} \min_{\mathrm{a}} E(\mathbf{m}, \mathbf{a}) = \min_{\mathrm{a}} \min_{\mathrm{m}} E(\mathbf{m}, \mathbf{a}) \tag{1.35}$$

Magnetostatic energy - two representations

1. We define n - dimensional Fourier transforms as

$$\mathcal{F}(f)(\mathbf{k}) \equiv \hat{f}(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{r}) e^{-i\mathrm{r}\cdot\mathrm{k}}\, d^n\mathbf{r}. \tag{1.36}$$

Using equation for magnetostatic potential u we obtain $\hat{u}(\mathbf{k}) = \frac{\widehat{\mathrm{m}}(\mathrm{k})\cdot\mathrm{k}}{|\mathrm{k}|^2}$

$$\int_{\mathbb{R}^3} |\nabla u|^2\, d^3\mathbf{r} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\widehat{\nabla u}|^2\, d^3\mathbf{k} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{|\mathbf{k}\cdot\widehat{\mathbf{m}}(\mathbf{k})|^2}{|\mathbf{k}|^2}\, d^3\mathbf{k}. \tag{1.37}$$

2. Energy can be written using $u(\mathbf{r}) = -\frac{1}{4\pi}\int_\Omega \frac{\nabla\cdot\mathrm{m}(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|} + \frac{1}{4\pi}\int_{\partial\Omega} \frac{(\mathrm{n}\cdot\mathrm{m})(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|}$ as

$$\begin{aligned} E_{ms}(\mathbf{m}) \equiv \frac{1}{2}\int_\Omega \nabla u\cdot\mathbf{m} &= \int_\Omega\int_\Omega \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\,\nabla\cdot\mathbf{m}(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|} \\ &+ \int_{\partial\Omega}\int_{\partial\Omega} \frac{(\mathbf{n}\cdot\mathbf{m})(\mathbf{r})\,(\mathbf{n}\cdot\mathbf{m})(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|} - 2\int_\Omega\int_{\partial\Omega} \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\,(\mathbf{n}\cdot\mathbf{m})(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|} \end{aligned} \tag{1.38}$$

Ferromagnetic thin films

We consider the following thin film domain

$$\Omega_\varepsilon = \{(x,y,z) : z \in [0,\varepsilon], (x,y) \in \omega \subset \mathbb{R}^2\}, \tag{1.39}$$

where $0 < \varepsilon \ll 1$ and want to simplify magnetostatic energy.

Fact. Let $\bar{\mathbf{m}} = \frac{1}{\varepsilon}\int_0^\varepsilon \mathbf{m}(x,y,z)dz$ and let $\Delta\bar{u} = \nabla \cdot \bar{\mathbf{m}}$. Then the following inequality holds:

$$\left| \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} |\nabla \bar{u}|^2 \right| \leq C\varepsilon^{\frac{3}{2}} \|\partial_z \mathbf{m}\|_{L^2(\Omega_\varepsilon)}. \tag{1.40}$$

This allows us to remove dependence of $\mathbf{m}$ on z variable in magnetostatics. We can now use Fourier representation and after calculation obtain

$$\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 = \frac{\varepsilon}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{m_3}(\mathbf{k}')|^2 \, d^2\mathbf{k}' + \frac{\varepsilon}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{\mathbf{m}'}(\mathbf{k}')\cdot\mathbf{k}'|^2 \frac{1-\hat{\Gamma}_\varepsilon(|\mathbf{k}'|)}{|\mathbf{k}'|^2} \, d^2\mathbf{k}'$$
$$- \frac{\varepsilon}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{m_3}(\mathbf{k}')\mathbf{k}'|^2 \frac{1-\hat{\Gamma}_\varepsilon(|\mathbf{k}'|)}{|\mathbf{k}'|^2} \, d^2\mathbf{k}', \tag{1.41}$$

where

$$\hat{\Gamma}_\varepsilon(|\mathbf{k}'|) = \frac{1-e^{-\varepsilon|\mathbf{k}'|}}{\varepsilon|\mathbf{k}'|} \sim 1 - \frac{\varepsilon|\mathbf{k}'|}{2} \tag{1.42}$$

Ferromagnetic thin films: Gioia-James regime

We have the following energy

$$E(\mathbf{m}) = \alpha \int_{\Omega_\varepsilon} |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r + \int_{\mathbb{R}^3} |\nabla u|^2 \tag{1.43}$$

and want to find its Γ-limit as $\varepsilon \to 0$.
It is clear using DCT that if $m_{3,\varepsilon} \to m_3$ in $L^2(\omega; \mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^2} |\widehat{m_{3,\varepsilon}}(\mathbf{k}')\mathbf{k}'|^2 \frac{1 - \hat{\Gamma}_\varepsilon(|\mathbf{k}'|)}{|\mathbf{k}'|^2} \, d^2\mathbf{k}' \to 0. \tag{1.44}$$

After rescaling domain in z-variable to
$\Omega = \{(x, y, z) : z \in [0, 1], (x, y) \in \omega \subset \mathbb{R}^2\}$ we obtain

$$E(\mathbf{m}) \sim \varepsilon\alpha \int_{\Omega} |\nabla' \mathbf{m}|^2 + \frac{1}{\varepsilon^2} |\partial_z \mathbf{m}|^2 + \varepsilon \int_{\omega} m_3^2 + o(\varepsilon) \tag{1.45}$$

Therefore, we can show (Gioia, James, *PRSA*, (1997))

$$\frac{1}{\varepsilon} E(\mathbf{m}_\varepsilon) \to \alpha \int_{\omega} |\nabla \mathbf{m}|^2 + \int_{\omega} m_3^2 \tag{1.46}$$

Ferromagnetic thin films: Kohn-S regime

We have the following energy

$$E(\mathbf{m}) = \alpha\varepsilon|\ln\varepsilon| \int_{\Omega_\varepsilon} |\nabla \mathbf{m}(\mathbf{r})|^2 \, d^2 r + \int_{\mathbb{R}^3} |\nabla u|^2 \tag{1.47}$$

and want to find its Γ-limit as $\varepsilon \to 0$. Rescaling domain in z-variable we obtain

$$\frac{1}{\varepsilon^2|\ln\varepsilon|} E(\mathbf{m}_\varepsilon) = \alpha \int_{\Omega} |\nabla' \mathbf{m}_\varepsilon|^2 + \frac{1}{\varepsilon^2}|\partial_z \mathbf{m}_\varepsilon|^2 + \frac{1}{\varepsilon^2|\ln\varepsilon|} \int_{\mathbb{R}^3} |\nabla u|^2 \tag{1.48}$$

We can show that

$$\frac{1}{\varepsilon^2|\ln\varepsilon|} \int_{\mathbb{R}^3} |\nabla u|^2 \sim \frac{1}{\varepsilon|\ln\varepsilon|} \int_{\omega} m_3^2 + \frac{1}{2\pi} \int_{\partial\omega} (\mathbf{m}_\varepsilon \cdot \mathbf{n})^2 \tag{1.49}$$

And hence (Kohn, S, *ARMA*, (2005))

$$\frac{1}{\varepsilon^2|\ln\varepsilon|} E(\mathbf{m}_\varepsilon) \to \alpha \int_{\omega} |\nabla \mathbf{m}|^2 + \frac{1}{2\pi} \int_{\partial\omega} (\mathbf{m} \cdot \mathbf{n})^2, \tag{1.50}$$

where $\mathbf{m} = \mathbf{m}(x, y)$ and $|\mathbf{m}| = 1$.