Introduction to Mathematical Micromagnetics

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Lecture 3. In-plane thin films: informal energy

In general, we have the following informal thin-film energy approximation

$$E(\mathbf{m}) \sim A_{ex} \int_\omega |\nabla \mathbf{m}|^2 + \int_\omega m_3^2 + \frac{\varepsilon |\ln \varepsilon|}{2\pi} \int_{\partial\omega} (\mathbf{m}' \cdot \mathbf{n})^2 \qquad (1.1)$$

$$+ \varepsilon \int_\omega \int_\omega \frac{\nabla \cdot \mathbf{m}'(\mathbf{r}) \, \nabla \cdot \mathbf{m}'(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \qquad (+ \text{ Field and Anisotropy})$$

Here $\omega \subset \mathbb{R}^2$, $\mathbf{m} = \mathbf{m}(x, y) \in \mathbb{S}^2$, $\mathbf{m}' = (m_1, m_2)$, $\varepsilon \ll 1$ is rescaled thickness.

There are multiple regimes depending on parameters

- Almost constant patterns for large A_{ex};
- In-plane configurations for smaller A_{ex};
- Vortices and cross-tie walls

In-plane thin films: shape anisotropy

- If $A_{ex} \gg \varepsilon|\ln\varepsilon|$ then $\mathbf{m} \in \mathbb{S}^1$ and (Carbou, *M3AS*, (2001))

$$E(\mathbf{m}) \sim \frac{1}{2\pi}\int_{\partial\omega}(\mathbf{m}\cdot\mathbf{n})^2 \quad (+\text{ FA}). \tag{1.2}$$

This is *shape anisotropy* and minimizers are not difficult to find.

For instance, for rectangular domain $\omega = [0,a]\times[0,b]$, $a > b$ we have $E(\mathbf{m}) = \frac{1}{2\pi}(a^2 m_2^2 + b^2 m_1^2)$ and minimizer is $\mathbf{m} = \pm(1,0)$.

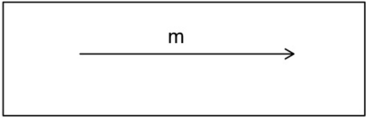


Figure: Magnetization in a rectangle.

In-plane thin films: configurational anisotropy

When $a = b$ (or, in general, for symmetric domains) there is a degeneracy leading to any in-plane **m** being a minimizer.
To resolve this degeneracy one has to keep exchange energy and study the energy

$$E(\mathbf{m}) = \frac{A_{ex}}{\varepsilon|\ln\varepsilon|}\int_{\omega}|\nabla\mathbf{m}|^2 + \frac{1}{2\pi}\int_{\partial\omega}(\mathbf{m}\cdot\mathbf{n})^2. \quad (1.3)$$

This leads to *configurational anisotropy* effect and necessity to perform Γ-expansion of the energy. For squares the degeneracy is broken at the next order leading to four minimizers $\mathbf{m} = \frac{1}{\sqrt{2}}(\pm 1, \pm 1)$ (Cowburn, Adeyeye, Welland, *PRL*, (1998); S, *PRSA*, (2010))

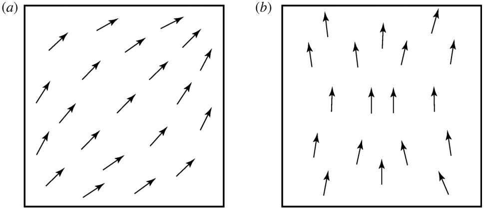


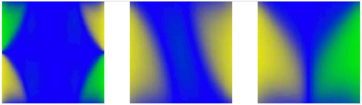
Figure: (a) Leaf state; (b) Flower state

In-plane thin films: S-state, C-state

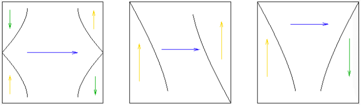
More minimal states appear when exchange and shape anisotropy compete

- If $A_{ex} \sim \alpha\varepsilon|\ln\varepsilon|$ then $\mathbf{m} \in H^1(\omega;\mathbb{S}^1)$ (Kohn, S, *ARMA*, (2005))

$$E(\mathbf{m}) = \alpha \int_\omega |\nabla \mathbf{m}|^2 + \frac{1}{2\pi}\int_{\partial\omega} (\mathbf{m}\cdot\mathbf{n})^2 \quad (+\ \mathrm{FA}). \qquad (1.4)$$



(a) Uniform state.
(b) S state.
(c) C State



(d) Sketch of Uniform state.
(e) Sketch of S state.
(f) Sketch of C state.

In-plane thin films: vortices

For simplicity we take $A_{ex} = \alpha\varepsilon$ then (Moser, *ARMA*, (2004))

$$E(\mathbf{m}) \sim \alpha \int_\omega |\nabla \mathbf{m}|^2 + \frac{1}{\varepsilon}\int_\omega m_3^2 + \frac{|\ln \varepsilon|}{2\pi}\int_{\partial\omega} (\mathbf{m}\cdot\mathbf{n})^2 + \int_\omega\int_\omega \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\,\nabla\cdot\mathbf{m}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

- We fix $\mathbf{m} = (m_1, m_2)$ to be in-plane and then we have formation of boundary vortices and energy becomes

$$E(\mathbf{m}_\varepsilon) \sim \left|\ln \frac{\ln\varepsilon}{2\pi\alpha}\right| \frac{\pi}{2}\sum_i d_i^2 + W(a_i, d_i) \qquad (1.5)$$

  where W is a renormalized energy depending on location of vortices $a_i \in \partial\omega$ are their degrees $d_i \in \mathbb{N}$ (Kurzke, *CVPDE*, (2006))

- We fix $\mathbf{m}\cdot\mathbf{n} = 0$ on $\partial\omega$, allow $m_3 \neq 0$ and then observe formation of interior vortices. The energy becomes

$$E(\mathbf{m}_\varepsilon) \sim \left|\ln \frac{1}{\varepsilon\alpha}\right| \pi\sum_i d_i^2 + W(a_i, d_i) \qquad (1.6)$$

  where W is a renormalized energy depending on location of vortices $a_i \in \omega$ are their degrees $d_i \in \mathbb{N}$.

In-plane thin films: vortices

We have the following energy

$$E(\mathbf{m}) \sim \alpha \int_\omega |\nabla \mathbf{m}|^2 + \frac{1}{\varepsilon} \int_\omega m_3^2 + \frac{|\ln \varepsilon|}{2\pi} \int_{\partial\omega} (\mathbf{m} \cdot \mathbf{n})^2 + \int_\omega \int_\omega \frac{\nabla \cdot \mathbf{m}(\mathbf{r}) \, \nabla \cdot \mathbf{m}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

"Mathematically" we prefer boundary vortex. However, $\varepsilon \sim 10^{-2}$ and hence boundary vortex competes with interior vortex (Ignat, Kurzke, *JFA*, (2023))

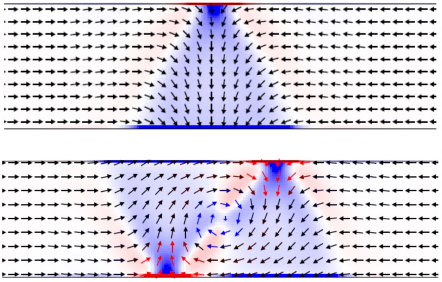


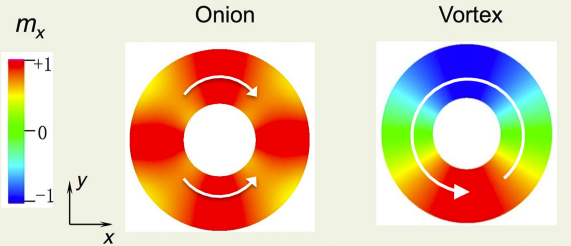
Figure: Transverse (up) and vortex (down) domain walls

In-plane thin films: Onion rings

- If $A_{ex} \sim \alpha\varepsilon$ and ω is a ring then meaningful energy is

$$E(\mathbf{m}) = \alpha \int_\omega |\nabla \mathbf{m}|^2 + \frac{|\ln \varepsilon|}{2\pi} \int_{\partial\omega} (\mathbf{m}\cdot\mathbf{n})^2 + \int_\omega \int_\omega \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\,\nabla\cdot\mathbf{m}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

Since the energy of a boundary vortex $\sim |\ln\ln\varepsilon|$ and $\varepsilon \sim 10^{-2}$, there is a competition between "almost constant" *onion state* and *vortex state* depending on α.



In plane thin-films: Aviles-Giga energy

If we take $A_{ex} \ll \varepsilon$ then $\nabla \cdot \mathbf{m} \sim 0$, $\mathbf{m} \sim (m_1, m_2, 0)$ and $(\delta \to 0)$

$$E(\mathbf{m}) \sim \delta \int_\omega |\nabla \mathbf{m}|^2 + \frac{1}{\delta} \|\nabla \cdot \mathbf{m}\|^2_{H^{-\frac{1}{2}}} + \frac{1}{\varepsilon\delta} \int_\omega m_3^2 \qquad (1.7)$$

We can link it to geometric problems through eikonal equation

$$\mathbf{m} = (m_1, m_2) = \nabla^\perp v, \quad |\nabla^\perp v| = 1 \qquad (1.8)$$

or look at the Aviles-Giga functional as a relaxed version (Ambrosio, De Lellis, Mantegazza, *CVPDE*, (1999); Jin, Kohn, *JNLS*, (2000))

$$E(\mathbf{m}) \sim \delta \int_\omega |\nabla(\nabla^\perp v)|^2 + \frac{1}{\delta} \int_\omega (1 - |\nabla^\perp v|^2)^2 \qquad (1.9)$$

For instance, cross-tie walls are observed in this regime (Alouges, Riviere, Serfaty, *ESAIM: COCV*, (2002)).

In plane thin-films: cross-tie walls

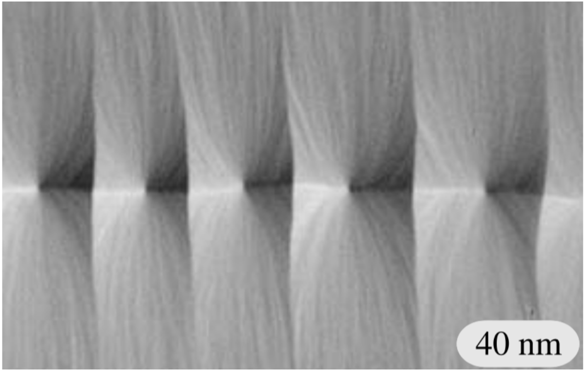


Figure: Cross-tie walls (Hubert, Schäfer, (1998))

Ultra-thin films with perpendicular magnetic anisotropy

We recall that magnetostatic energy for $\mathbf{m}(x, y)$ is

$$\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 \sim \frac{\varepsilon}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{m_3}(\mathbf{k}')|^2 \, d^2\mathbf{k}' + \frac{\varepsilon^2}{2(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{\mathbf{m}'}(\mathbf{k}') \cdot \mathbf{k}'|^2 \frac{1}{|\mathbf{k}'|} \, d^2\mathbf{k}' - \frac{\varepsilon^2}{2(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{m_3}(\mathbf{k}')\mathbf{k}'|^2 \frac{1}{|\mathbf{k}'|} \, d^2\mathbf{k}'. \quad (1.10)$$

We can introduce a new 2D energy for ultra thin films

$$E_\varepsilon(\mathbf{m}) = \int_{\mathbb{R}^2} \eta_\varepsilon^2 |\nabla \mathbf{m}|^2 + (Q-1)\eta_\varepsilon^2 (1 - m_3^2) d^2 r + \frac{\gamma}{2|\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot (\eta_\varepsilon \mathbf{m})(\mathbf{r}) \nabla \cdot (\eta_\varepsilon \mathbf{m})(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' - \frac{\gamma}{4|\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\eta_\varepsilon(\mathbf{r}) m_3(\mathbf{r}) - \eta_\varepsilon(\mathbf{r}') m_3(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r', \quad (1.11)$$

where η_ε is a cut-off function with $\eta = 1$ in $\Omega \subset \mathbb{R}^2$ and $\eta = 0$ in $\Omega_\varepsilon = \Omega \cup O_\varepsilon$.

Ultra-thin films with perpendicular magnetic anisotropy

Theorem 1

Let γ be a constant. Then as $\varepsilon \to 0$ the energy E_ε Γ-converges to E_0 in L^2

$$E_0(\mathbf{m}) = \int_\Omega |\nabla \mathbf{m}|^2 + (Q-1)(1-m_3^2) d^2 r + \gamma \int_{\partial\Omega} \left((\mathbf{m}' \cdot \mathbf{n})^2 - m_3^2 \right) d\mathcal{H}^1(\mathbf{r}')$$

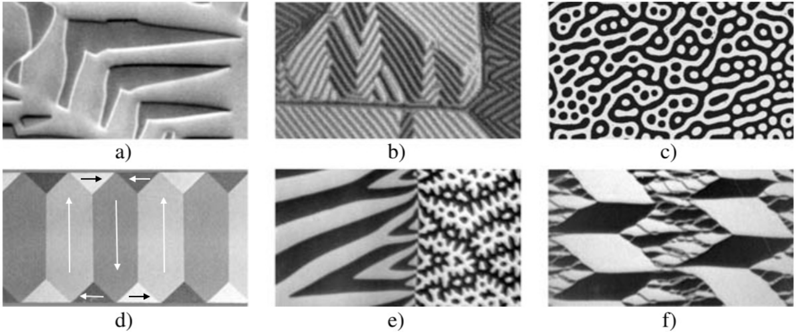
Theorem 2

Let $\gamma = \nu |\ln \varepsilon|$. Then as $\varepsilon \to 0$ the energy E_ε Γ-converges to E_0 in L^2

$$\begin{aligned} E_0(\mathbf{m}) = & \int_\Omega |\nabla \mathbf{m}|^2 + (Q-1)(1-m_3^2) d^2 r + \nu \int_\Omega \mathbf{b} \cdot \nabla m_3 \, d^2 r \\ & + \frac{\nu}{2} \int_\Omega \int_\Omega \frac{\nabla \cdot \mathbf{m}'(\mathbf{r}) \nabla \cdot \mathbf{m}'(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' \\ & - \frac{\nu}{4} \int_\Omega \int_\Omega \frac{(m_3(\mathbf{r}) - m_3(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r', \end{aligned} \tag{1.12}$$

where $\mathbf{m} = \pm \mathbf{e}_3$ *on* $\partial\Omega$ *and* $\mathbf{b}(\mathbf{r}) = \int_{\partial\Omega} \frac{m_3(\mathbf{r}')\mathbf{n}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\mathcal{H}^1(\mathbf{r}')$.

Films with PMA



Lecture 3. Films with perpendicular anisotropy.

We want to understand some basic patterns in PMA films. For simplicity we impose **periodic boundary conditions** on lateral boundary of the film

$$\Omega = \mathbb{R}^2 \times \left(-\frac{L}{2}, \frac{L}{2}\right), \tag{1.13}$$

with a periodic cell $D = \left(-\frac{p_1}{2}, \frac{p_1}{2}\right) \times \left(-\frac{p_2}{2}, \frac{p_2}{2}\right)$.
We want to find the total energy per cell

$$E(\mathbf{m}) = \frac{1}{|D|} \int_{D\times(-L/2,L/2)} \varepsilon^2 |\nabla \mathbf{m}|^2 + Q(1 - m_3^2)\, d^3r + \frac{1}{|D|} \int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r$$

We first compute the magnetostatic energy per cell

$$\frac{1}{|D|} \int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \int_{\mathbb{R}} \frac{\left|\frac{n_1}{p_1}\hat{m}_1 + \frac{n_2}{p_2}\hat{m}_2 + \xi\hat{m}_3\right|^2}{\frac{n_1^2}{p_1^2} + \frac{n_2^2}{p_2^2} + \xi^2}\, d\xi, \tag{1.14}$$

where

$$\hat{\mathbf{m}}(n_1, n_2, \xi) = \frac{1}{p_1 p_2} \int_{\mathbb{R}} e^{-i2\pi z\xi} \int_Q \mathbf{m}(x, y, z) e^{-i2\pi\left(\frac{x}{p_1}n_1 + \frac{y}{p_2}n_2\right)}\, dx\, dy\, dz.$$

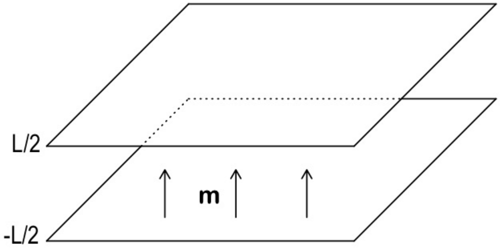
Thin films with PMA – constant magnetization

We first compute magnetostatic energy for $\mathbf{m} = \mathbf{e}_3$

$$\hat{\mathbf{m}}(n_1, n_2, \xi) = \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-i2\pi z\xi}\, dz = \begin{cases} \frac{\sin(\pi L\xi)}{\pi\xi}\mathbf{e}_3 & \text{when } n_1 = n_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E(\mathbf{m}) = \frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r = \frac{1}{\pi^2}\int_{\mathbb{R}} \frac{\sin^2(\pi L\xi)}{\xi^2}\, d\xi = L. \tag{1.15}$$



Thin films with PMA – chess board pattern

Assume we have an ultra-thin film with $\mathbf{m}$ being independent of z variable. We define the periodic *chess-board pattern* as

$$\mathbf{m}(x,y) = \begin{cases} -\mathbf{e}_3 & (x,y) \in \left(-\frac{p_1}{2}, 0\right) \times \left(-\frac{p_2}{2}, 0\right) \cup \left(0, \frac{p_1}{2}\right) \times \left(0, \frac{p_2}{2}\right) \\ \mathbf{e}_3 & (x,y) \in \left(-\frac{p_1}{2}, 0\right) \times \left(0, \frac{p_2}{2}\right) \cup \left(0, \frac{p_1}{2}\right) \times \left(-\frac{p_2}{2}, 0\right) \end{cases} \quad (1.16)$$

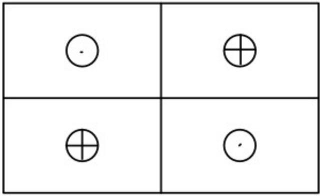


Figure: Top view of thin film

Therefore we have

$$\hat{\mathbf{m}}(n_1, n_2, \xi) = \begin{cases} \frac{4\sin(\pi L \xi)}{\pi^3 \xi n_1 n_2} \mathbf{e}_3 & \text{when } n_1 \cdot n_2 \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \quad (1.17)$$

Magnetostatics – chess board pattern

The magnetostatic energy is

$$\frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2 \, d^3 r = \frac{16}{\pi^5} \sum_{(n_1,n_2)\in\mathbb{Z}_{odd}\times\mathbb{Z}_{odd}} \frac{1-e^{-2\pi L\sqrt{\frac{n_1^2}{p_1^2}+\frac{n_2^2}{p_2^2}}}}{n_1^2 n_2^2 \sqrt{\frac{n_1^2}{p_1^2}+\frac{n_2^2}{p_2^2}}} \tag{1.18}$$

We assume $p_1 = p_2 \equiv p$ and $\nu = \frac{L}{p} \ll 1$ then we obtain (Kaplan, Gehring, *JMMM*, (1993))

$$\frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2 \, d^3 r \sim \frac{L}{\nu} \sum_{(n_1,n_2)\in\mathbb{Z}_{odd}\times\mathbb{Z}_{odd}} \frac{1-e^{-2\pi\nu\sqrt{n_1^2+n_2^2}}}{n_1^2 n_2^2 \sqrt{n_1^2+n_2^2}} \tag{1.19}$$

$$\sim L\left(1-\alpha_c \nu + \frac{8}{\pi}\nu \ln \nu\right) \tag{1.20}$$

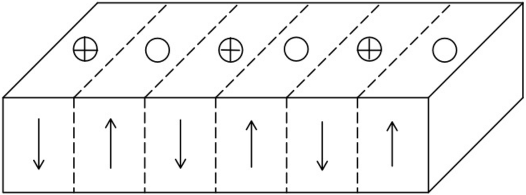
Thin films with PMA – stripes pattern

We know magnetostatic energy of chess-board. We can assume $p_1 \equiv p$, $p_2 = \infty$ and $\nu = \frac{L}{p}$ ($\nu_{KG} = \frac{2L}{p} = 2\nu$) then we obtain (Kaplan, Gehring, *JMMM*, (1993))

$$\frac{1}{|D|} \int_{\mathbb{R}\times D} |\nabla u|^2 \, d^3 r = \frac{2}{\pi^3} \frac{L}{\nu} \sum_{n \in \mathbb{Z}_{odd}} \frac{1 - e^{-2\pi\nu n}}{n^3} \tag{1.21}$$

$$\sim \frac{L}{2} \left(1 - \alpha_s \nu + \frac{4}{\pi} \nu \ln \nu \right) \tag{1.22}$$

where $\alpha_s = \frac{2}{\pi}(3 - 2\ln\pi)$.



Films with PMA – simplified energy.

We want to find the total energy

$$E(\mathbf{m}) = \frac{1}{2|D|}\int_{D\times(-L/2,L/2)} \varepsilon^2|\nabla \mathbf{m}|^2 + Q(1-m_3^2)\,d^3r + \frac{1}{2|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\,d^3r$$

Clearly this pattern will have infinite exchange energy but we can always approximate it by smooth transitions (Bloch walls) and the total energy is

$$E(\mathbf{m}) \sim \frac{\varepsilon\sqrt{Q}}{|D|}\int_{D\times(-L/2,L/2)} |\nabla \mathbf{m}| + \frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\,d^3r.$$

In general, we can concentrate on the following energy to understand all terms in more complicated patterns with sharp transitions

$$E(\mathbf{m}) = \frac{\varepsilon}{|D|}\int_{D\times(-L/2,L/2)} |\nabla \mathbf{m}|\,d^3r + \frac{Q}{|D|}\int_{D\times(-L/2,L/2)} (1-m_3^2)\,d^3r + \frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\,d^3r.$$

Thin films with PMA – optimal period.

Let us now compute the energy of chess board and stripes patterns

- **Chess-board.** It follows that (recalling $\nu = L/p$)

$$E(\mathbf{m}) \sim \varepsilon\sqrt{Q}\nu + L\left(1 - \alpha_c\nu + \frac{4}{\pi}\nu\ln\nu\right)$$

Minimizing in ν we obtain

$$\nu \sim e^{\frac{-\pi}{4L}\left(\varepsilon\sqrt{Q}+L\left(\frac{4}{\pi}-\alpha_c\right)\right)}$$

and hence the optimal period p.

- **Stripes.** Similar calculation for stripes gives us

$$\nu \sim e^{\frac{-\pi}{2L}\left(\varepsilon\sqrt{Q}+L\left(\frac{2}{\pi}-\alpha_s\right)\right)}$$

There is a set of rigorous works on these problems (e.g Choksi, Kohn, Otto, *CMP*, (1999); Otto, Viehmann, *CVPDE*, (2010); Knüpfer, Muratov, Nolte, *ARMA*, (2019); Brietzke, Knüpfer, *CVPDE*, (2023))