Introduction to Mathematical Micromagnetics

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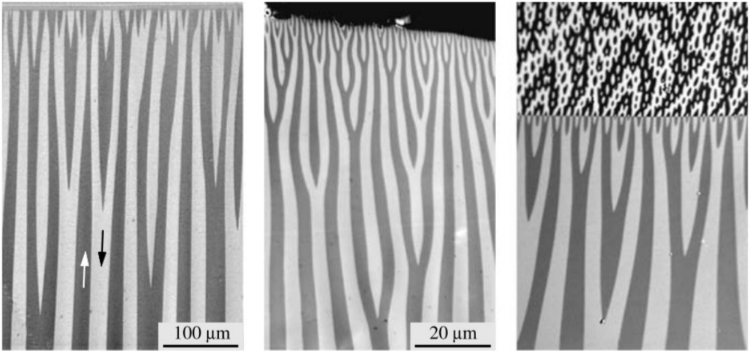
Lecture 4. Thick films with PMA.

We want to understand magnetization branching in thick films (following Choksi, Kohn, *CPAM*, (1998); Choksi, Kohn, Otto, *CMP*, (1999); Desimone, Kohn, Müller, Otto, (2006))



a) b) c)

Thick films with PMA – no z dependence.

- Do we loose if we don't branch and **m** is independent on z?

Let us compute the *upper bound on energy* of thick films, assuming **m** is independent of z. To allow for sharp transitions

$$E(\mathbf{m}) = \frac{\varepsilon}{|D|}\int_{D\times(-L/2,L/2)} |\nabla \mathbf{m}|\, d^3r + \frac{Q}{|D|}\int_{D\times(-L/2,L/2)} (1-m_3^2)\, d^3r + \frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r.$$

For chess-board pattern (D has side p) with $x = L/p \gg 1$ we have

$$\frac{1}{2|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r \sim p \sum_{(n_1,n_2)\in\mathbb{Z}_{odd}\times\mathbb{Z}_{odd}} \frac{1-e^{-2\pi x\sqrt{n_1^2+n_2^2}}}{n_1^2 n_2^2\sqrt{n_1^2+n_2^2}} \sim p \qquad (1.1)$$

The total energy is

$$E(\mathbf{m}) \sim \varepsilon L/p + p \sim \sqrt{\varepsilon L}$$

If we consider stripes or Kittel pattern we still obtain $E(\mathbf{m}) \sim \sqrt{\varepsilon L}$.

Energy lower bound – no z dependence.

We now want to find a lower bound on this energy, assume $Q \gtrsim 1$, $\sqrt{\varepsilon L} < 1$)

$$E(\mathbf{m}) = \frac{\varepsilon}{|D|}\int_{D\times(-L/2,L/2)} |\nabla \mathbf{m}|\, d^3r + \frac{Q}{|D|}\int_{D\times(-L/2,L/2)} (1 - m_3^2)\, d^3r + \frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r.$$

We recall that (assuming for simplicity unit periodic cell)

$$\frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \int_{\mathbb{R}} \frac{|\overbrace{n_1\hat{m}_1 + n_2\hat{m}_2}^{a} + \overbrace{\xi\hat{m}_3}^{b}|^2}{n_1^2 + n_2^2 + \xi^2}\, d\xi, \qquad (1.2)$$

and hence using $(a+b)^2 \geq \frac{1}{2}b^2 - a^2$ and controlling $-a^2$ with anisotropy

$$\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r + \int_{D\times(-L/2,L/2)} m_1^2 + m_2^2 \gtrsim \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \int_{\mathbb{R}} \frac{|\xi\hat{m}_3|^2}{n_1^2 + n_2^2 + \xi^2}\, d\xi$$

Energy lower bound – no z dependence.

We can now write this energy bound

$$E(\mathbf{m}) \gtrsim \varepsilon \int_{D\times(-L/2,L/2)} |\nabla' m_3|\, d^3r + Q \int_{D\times(-L/2,L/2)} (1-m_3^2)\, d^3r$$
$$+ \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \int_{\mathbb{R}} \frac{|\xi \hat{m}_3|^2}{n_1^2+n_2^2+\xi^2}\, d\xi$$

Assuming $\mathbf{m}(x,y)$, then (as for thin films) we explicitly find $\hat{m}_3(\mathbf{n})$ and

$$E(\mathbf{m}) \gtrsim \varepsilon L \int_D |\nabla' m_3|\, d^2r + QL \int_D (1-m_3^2)\, d^2r$$
$$+ \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \int_{\mathbb{R}} \frac{|\hat{m}_3(\mathbf{n})|^2 \sin^2(2\pi L\xi)}{n_1^2+n_2^2+\xi^2}\, d\xi$$

After integrating in ξ we arrive at

$$E(\mathbf{m}) \gtrsim \varepsilon L \sup_{x'\in D} |m_3| \int_D |\nabla' m_3|\, d^2r + QL \int_D (1-m_3^2)\, d^2r$$
$$+ L \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |\hat{m}_3(\mathbf{n})|^2 \min\left\{1, \frac{1}{L|\mathbf{n}|}\right\}$$

Energy lower bound – no z dependence.

Now we want to obtain a bound on the following sum

$$S = \varepsilon L \sup_{x' \in D} |m_3| \int_D |\nabla' m_3| + L \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |\hat{m}_3(\mathbf{n})|^2 \min\left\{1, \frac{1}{L|\mathbf{n}|}\right\}$$

$$\gtrsim \sqrt{\varepsilon L}\left(\sqrt{\varepsilon L} \sup_{x' \in D} |m_3| \int_D |\nabla' m_3| + \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |\hat{m}_3(\mathbf{n})|^2 \min\left\{1, \frac{1}{\sqrt{\varepsilon L}|\mathbf{n}|}\right\}\right)$$

Now we have the following estimate

$$2 \sup_{x' \in D} |m_3| \int_D |\nabla' m_3 \cdot y'| \geq \int_D |m_3(x'+y') - m_3(x')|^2 \tag{1.3}$$

$$= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} 4\sin^2\left(\frac{\mathbf{n}\cdot y'}{2}\right) |\hat{\mathbf{m}}(\mathbf{n})|^2 \tag{1.4}$$

Energy lower bound – no z dependence.

It is possible to show that (taking $\alpha = \sqrt{\varepsilon L}$ and integrating over $|y'| = \alpha$)

$$\alpha \sup_{x' \in D} |m_3| \int_D |\nabla' m_3| \gtrsim \sum_{|\mathrm{n}| \geq \alpha^{-1}} |\hat{m}_3(\mathbf{n})|^2 \tag{1.5}$$

On the other hand we have

$$\sum_{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}} |\hat{m}_3(\mathbf{n})|^2 \min \left\{ 1, \frac{1}{\sqrt{\varepsilon L} |\mathbf{n}|} \right\} \gtrsim \sum_{|\mathrm{n}| \leq \alpha^{-1}} |\hat{m}_3(\mathbf{n})|^2 \tag{1.6}$$

Combining above estimates we arrive at

$$S \gtrsim \sqrt{\varepsilon L} \sum_{\mathrm{n}} |\hat{m}_3(\mathbf{n})|^2$$

Therefore the energy

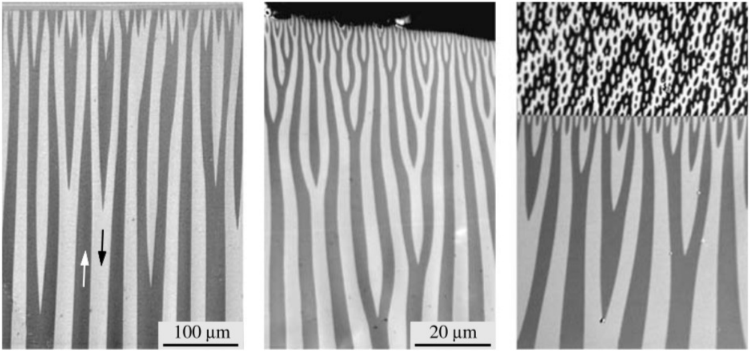
$$E(\mathbf{m}) \gtrsim \sqrt{\varepsilon L} \int_D m_3^2 \, d^2 r + QL \int_D (1 - m_3^2) \, d^2 r \gtrsim \sqrt{\varepsilon L}.$$

This is the matching lower bound for the micromagnetic energy of thick film without z dependence.

Thick films with PMA.

- Assumption on **m** being independent on z is not natural for thick films.

We should be able to decrease the energy by introducing z dependence.



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Landau construction and divergence free fields.

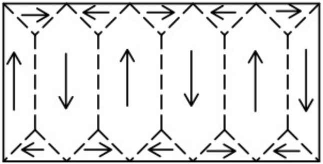

Figure: Landau pattern

We want to show that magnetostatic energy for this pattern vanishes. It is enough to have $\int_\Omega \mathbf{m}\cdot\nabla\phi = 0$ for all $\phi \in H_0^1(\Omega)$ and $\mathbf{m}\cdot\mathbf{n} = 0$ on $\partial\Omega$.

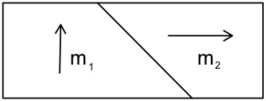

Figure: Div free $\mathbf{m}$ inside domain: $\mathbf{m}_1\cdot\mathbf{n} = \mathbf{m}_2\cdot\mathbf{n}$ at transition

Energy bound for Landau pattern.

We can consider Landau pattern and obtain energy bound for

$$E(\mathbf{m}) = \frac{\varepsilon}{|D|}\int_{D\times(-L/2,L/2)} |\nabla \mathbf{m}|\, d^3r + \frac{Q}{|D|}\int_{D\times(-L/2,L/2)} (1-m_3^2)\, d^3r + \frac{1}{|D|}\int_{\mathbb{R}\times D} |\nabla u|^2\, d^3r.$$

$$E_{ms}(\mathbf{m}) = 0, \quad E_{an}(\mathbf{m}) \sim Qp, \quad E_{tr}(\mathbf{m}) \sim \varepsilon \frac{L}{p}$$

The optimal $p \sim \varepsilon^{1/2}L^{1/2}/Q^{1/2}$ and total energy is

$$E(\mathbf{m}) \sim Q^{1/2}\varepsilon^{1/2}L^{1/2}$$

So we have a similar scaling as for z independent patterns ($Q \sim 1$)

$$E(\mathbf{m}) \sim \sqrt{\varepsilon L}$$

Privorotski construction.

We do the following 2D construction (Privorotski, *Rep. Prog. Phys.*, (1972))

- In the middle of domain we take $\mathbf{m} = \pm\mathbf{e}_3$ in strips of width a;
- Moving towards the boundary we refine the pattern as shown in figure

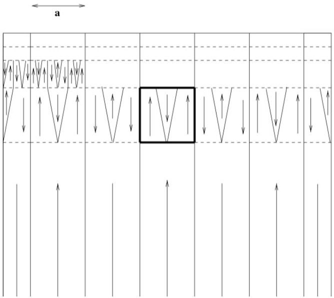

Figure: Branching pattern (upper side)

Privorotski construction.

Let us explain the construction inside a cell

- Cell is split in three regions: in the middle region $\mathbf{m} = -e_3$, in the right side region $\mathbf{m}$ is tangent to circles centered at O or O'; $R_n = |OP|$.
- Inside cell $\nabla \cdot \mathbf{m} = 0$ and it defines boundaries between cells ($\mathbf{m} \cdot \mathbf{n}$ is continuous across the boundary): $r = R_n \sec^2(\theta/2)$, $r' = \frac{a_n}{6} \sec^2(\theta/2)$
- Continuity across OO' yields $\frac{a_n}{3} = R_n(\sec(\theta_n) - 1)$.

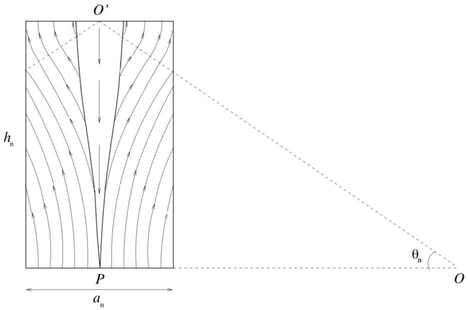

Figure: Privorotski cell at n-th level

Privorotski construction – cell energy.

• **Magnetostatic energy.** Inside the domain we have $\nabla \cdot \mathbf{m} = 0$ and only top and bottom boundaries contribute to stray field. If construction has N layers then magnetization at the top boundary oscillates with the period $p = \frac{a}{3^N}$, taking values $\pm \mathbf{e}_3$.

Therefore, we already know what the magnetostatic energy is from Kittel construction. Indeed, we have

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi \, d^3 r = \int_{\Omega} \mathbf{m} \cdot \nabla \phi \, d^3 r = \int_{\partial \Omega} \mathbf{m} \cdot \mathbf{n} \phi \, d^2 r \tag{1.7}$$

and since $\mathbf{m}$ at the boundary Ω of a simple Kittel pattern and Privorotski pattern (with attuned period) coincide, we deduce that

$$\frac{1}{2|Q|} \int_{\mathbb{R} \times Q} |\nabla u|^2 \, d^3 r = \frac{p}{2\pi^3} \sum_{n_1 \in \mathbb{Z}_{odd}} \frac{1 - e^{-2\pi L \frac{n_1}{p}}}{n_1^3} \sim \alpha \frac{a}{3^N}. \tag{1.8}$$

Privorotski construction – cell energy.

Anisotropy energy is defined as $E_a(\mathbf{m}) = Q\int (1 - m_3^2)\, d^2 r$. In the middle region of the cell magnetization is $\mathbf{m} = \pm \mathbf{e}_3$ – no contribution from this region. In the left and right regions magnetization $\mathbf{m} = (-\sin\theta, 0, \cos\theta)$ (as it is tangent to corresponding circles) and hence (for $\theta_n \ll 1$)

$$E_a^n(\mathbf{m}) = Q\int_0^{\theta_n} \sin^2\theta \left(\int_{(R_n - a_n/2)/\cos\theta}^{R_n/\cos^2(\theta/2)} r dr + \int_{a_n/(6\cos^2(\theta/2))}^{a_n/(2\cos\theta)} r dr \right) d\theta \sim C Q a_n^2 \theta_n$$

Surface energy is defined as $E_s(\mathbf{m}) = \varepsilon \int |\nabla \mathbf{m}|\, d^2 r$. We recall $R_n \sim \frac{2a_n}{3\theta_n^2}$ and $h_n \sim \frac{2a_n}{3\theta_n}$. Therefore

$$E_s^n(\mathbf{m}) \sim C\varepsilon h_n \sim C\frac{a_n \varepsilon}{\theta_n}$$

We combine surface and anisotropy energies and minimize in θ_n to obtain

$$E_s^n(\mathbf{m}) + E_a^n(\mathbf{m}) \sim C\frac{a_n\varepsilon}{\theta_n} + CQa_n^2\theta_n \sim \sqrt{\varepsilon Q}\, a_n^{\frac{3}{2}}, \quad \theta_n \sim \sqrt{\frac{\varepsilon}{a_n Q}}$$

Privorotski construction – total energy.

Total energy. We have $a_n = \frac{a}{3^n}$ and define $h = \sum_{n=1}^{N} h_n$. The surface energy of the middle "Kittel" region is $\sim C\frac{\varepsilon(L-h)}{a}$. We compute the total energy as

$$E(\mathbf{m}) \sim \frac{2\varepsilon(L-h)}{a} + \frac{1}{a}\sum_{n=1}^{N} 3^{n-1}\sqrt{\varepsilon Q} a_n^{\frac{3}{2}} + C\frac{a}{3^N} \leq C\frac{2\varepsilon L}{a} + C\sqrt{\varepsilon Q a}$$

$$\lesssim C Q^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}, \quad \text{with optimal } a \sim \frac{\varepsilon^{\frac{1}{3}}L^{\frac{2}{3}}}{Q^{\frac{1}{3}}}$$

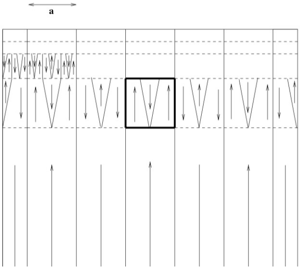

Figure: Branching pattern (upper side)

Energy lower bound.

Similarly to the z-independent case we have

$$E(\mathbf{m}) \gtrsim \varepsilon \int_{D\times(-L/2,L/2)} |\nabla' m_3|\, d^3 r + Q \int_{D\times(-L/2,L/2)} (1-m_3^2)\, d^3 r$$
$$+ \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \int_{\mathbb{R}} \frac{|\xi \hat{m}_3|^2}{n_1^2+n_2^2+\xi^2}\, d\xi$$

After calculation we arrive at

$$E(\mathbf{m}) \gtrsim \int_{-L/2}^{L/2} \Bigg(\varepsilon \sup_{x'\in D} |m_3| \int_D |\nabla' m_3|\, d^2 r + Q \int_D (1-m_3^2)\, d^2 r$$
$$+ \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |\hat{m}_3(\mathbf{n},z)|^2 \min\left\{1, \frac{1}{(L|\mathbf{n}|)^2}\right\} \Bigg)$$

Combining the first and the last two terms as before we obtain

$$S = \varepsilon \sup_{x'\in D} |m_3| \int_D |\nabla' m_3| + \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |\hat{m}_3(\mathbf{n},z)|^2 \min\left\{1, \frac{1}{(L|\mathbf{n}|)^2}\right\}$$
$$\gtrsim \frac{\varepsilon^{2/3}}{L^{2/3}} \left(\varepsilon^{1/3} L^{2/3} \sup_{x'\in D} |m_3| \int_D |\nabla' m_3| + \sum_{\mathbf{n}\in\mathbb{Z}\times\mathbb{Z}} |\hat{m}_3(\mathbf{n},z)|^2 \min\left\{1, \frac{1}{(\varepsilon^{1/3}L^{2/3}|\mathbf{n}|)^2}\right\} \right)$$

Energy lower bound.

Taking $\alpha = \varepsilon^{1/3} L^{2/3}$ and using as before

$$\alpha \sup_{x' \in D} |m_3| \int_D |\nabla' m_3| \gtrsim \sum_{|\mathrm{n}| \geq \alpha^{-1}} |\hat{m}_3(\mathbf{n})|^2 \tag{1.9}$$

and

$$\sum_{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}} |\hat{m}_3(\mathbf{n})|^2 \min \left\{ 1, \frac{1}{(\alpha |\mathbf{n}|)^2} \right\} \gtrsim \sum_{|\mathrm{n}| \leq \alpha^{-1}} |\hat{m}_3(\mathbf{n})|^2 \tag{1.10}$$

Combining above estimates we arrive at

$$S \gtrsim \frac{\varepsilon^{2/3}}{L^{2/3}} \sum_{\mathrm{n}} |\hat{m}_3(\mathbf{n})|^2$$

Therefore the energy

$$E(\mathbf{m}) \gtrsim \frac{\varepsilon^{2/3}}{L^{2/3}} \int_{D \times (-L/2.L/2)} m_3^2 \, d^2 r + Q \int_{D \times (-L/2, L/2)} (1 - m_3^2) \, d^2 r \gtrsim \varepsilon^{2/3} L^{1/3}.$$

This is the matching lower bound for the micromagnetic energy of thick film with z dependence.