Pattern formation: nonlinear dynamics and multiscale analysis in reaction-diffusion systems

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The aim of this course is to provide an introduction to the mathematical theory of pattern formation, namely the dynamical transition by which a physical system can evolve from a spatially uniform steady configuration towards a spatially structured state.

This course will present an introduction to the basic mathematical tools used to understand complex pattern dynamics in reaction-diffusion systems. It will mainly focus on two-component reaction-diffusion systems. The techniques by which pattern formation can be studied (both 'near onset' as well as 'far from equilibrium') will be presented in such a way that an explicit study of a given model can be set up.

After the introduction of the mechanism of destabilization leading to Turing patterns and the corresponding spectral stability analysis, we shall present the weakly nonlinear theory based on a multiple scales analysis and aimed to derive the amplitude equations, ordinary differential equations that describe the dynamics close to the bifurcation.

The occurrence of competing instabilities determining codimension-two bifurcations will also be treated. We will focus on the interaction between stationary and oscillatory instabilities that is responsible for the emergence of oscillating spatial structures; in particular Turing-Hopf bifurcations and subharmonic instabilities will be addressed.

The analysis of 'far from equilibrium' dynamics will include (i) transition layers and (ii) localized patterns. (i) Transition layers are physically relevant configurations observed in spatially inhomogeneous states which exhibit sharp interfaces at some specified location. Mesa-patterns, spikes and pulses are all examples of such structures. We shall show how, adopting appropriate matching asymptotic methods, they can be constructed as solutions of singularly perturbed reaction-diffusion-chemotaxis systems of biological interest. (ii) Localized structures generated by non-conservative systems (also named *dissipative solitons*) are found in presence of bistability between a spatially homogeneous state and a large amplitude periodic pattern. These states are organized in a characteristic snakes-and-ladders structure, displaying the so-called *homoclinic snaking phenomenon* as they approach the infinite-length spatially periodic pattern. The above issue will be addressed and the origin of the spatially localized patterns will be shown to be originated by a reversible $1:1$ resonance (Hamilton-Hopf bifurcation), whose normal form gives the explicit expression of the small-amplitude branch of homoclinic solutions.

Finally, particular attention will be devoted to the role played in the emergence of the bifurcations by the presence of nonlinear diffusion terms, usually not included in most of the *celebrated* model problems although often required for a realistic description of the phenomena. Cross-diffusion terms as well as chemotaxis effects, if on the one hand introduce severe mathematical complications, on the other hand can be responsible for the occurrence of complex spatio-temporal structures. In this respect, representative systems of mathematical biology as well as some newly derived models will be discussed.