PROGRAM OF THE COURSE

INSTABILITY AND BIFURCATION

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By the Principle of Least Action, physical systems governed by conservative forces typically assume energy-minimizing states. As such, these ground states are *stable* stationary points of the corresponding energy functional. On the other hand, *unstable* stationary points may also have very interesting features from a mathematical viewpoint, although (or, perhaps, because) they are harder to find in nature.

If the way energy is measured depends on a parameter, a family of stationary points may lose stability when that parameter crosses a certain threshold. Remarkably, this loss of stability creates a new branch of stationary points that splits from the family. This phenomenon was first explored by Poincaré, who called it a *bifurcation*, marking the dawn of a multifaceted theory with applications to Dynamical Systems, Analysis, PDEs, and, more recently, to Differential Geometry and Geometric Analysis.

In this course, we give an overview of classical results in variational Bifurcation Theory and some geometric applications, including multiplicity results for Geodesics, Constant Mean Curvature Surfaces, and the Yamabe problem. These are obtained exploiting the growing instability of families of trivial (often highly symmetric) solutions as they degenerate. The resulting bifurcating solutions are often less symmetric, and give rise to interesting examples where ground states need not be the most symmetric ones.

The course will be given in a series of 16 lectures of about two hours each, tentatively during the months of November and Deecember 2021.

Prerequisites. It will be assumed that the student have basic knowledge of Differential Geometry and Calculus on Manifolds.

COURSE PROGRAM

(1) Generalities on variational problems in Riemannian geometry
   - Riemannian manifolds, connections, curvature.
   - Geeodesics, Jacobi fields, conjugate points.
   - Fredholm operators, bilinear forms, index.
   - The Morse Index Theorem

(2) Notions of abstract bifurcation theory

Date: January 25, 2021.

- Bifurcation from a simple eigenvalue
- Global bifurcation
- Variational bifurcation via Morse index.
- Equivariant problems and symmetry breaking.

(3) Bifurcation for the geodesic variational problem.
- Fixed endpoints geodesics.
- The periodic case.
- Free endpoints.

(4) Lorentzian and semi-Riemannian geodesics
- Timelike and lightlike geodesics: an application to General Relativity.
- Maslov index
- Spectral flow
- Bifurcation at a nondegenerate conjugate point.

(5) The Yamabe problem
- The Yamabe problem and the elliptic nonlinear equation
- First and second variation of the Hilbert–Einstein functional
- Local rigidity
- Bifurcation in product manifolds and in Riemannian homogeneous submersions.
- A bifurcation problem for the singular Yamabe problem on spheres.

(6) Minimal submanifolds and Constant Mean Curvature hypersurfaces
- Extrinsic geometry: second fundamental form, minimal and CMC surfaces
- Bifurcation of Clifford tori.
- Minimal 2-spheres in ellipsoids.

BIBLIOGRAPHY

- R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in calculus of variations (Montecatini Terme, 1987), 120–154, Lecture Notes in Math., 1365, Springer, Berlin, 1989.
- J. Smoller, A. Wasserman, *Bifurcation and symmetry-breaking*. Invent. Math. 100 (1990), no. 1, 63–95.
- L. L. de Lima, P. Piccione, M. Zedda, *On bifurcation of solutions of the Yamabe problem in product manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 2, 261–277.
- P. Piccione, A. Portaluri, D. V. Tausk, *Spectral flow, Maslov index and bifurcation of semi-Riemannian geodesics*, Ann. Global Anal. Geom. 25 (2004), no. 2, 121–149.
- R. Giambò, F. Giannoni, P. Piccione, *Gravitational lensing in general relativity via bifurcation theory*, Nonlinearity 17 (2004), no. 1, 117–132.

- R. Bettiol, P. Piccione, *Delaunay–type hypersurfaces in cohomogeneity one manifolds*, Int. Math. Res. Not. IMRN 2016, no. 10, 3124–3162.
- R. Bettiol, P. Piccione, B. Santoro, *Bifurcation of periodic solutions to the singular Yamabe problem on spheres*, J. Differential Geom. 103 (2016), no. 2, 191–205.
- L. J. Alías, P. Piccione, *Bifurcation of constant mean curvature tori in Euclidean spheres* J. Geom. Anal. 23 (2013), no. 2, 677–708.
- L. Biliotti, M. A. Javaloyes and P. Piccione, *On the semi-Riemannian bumpy metric theorem*, J. Lond. Math. Soc. (2) 84(1) (2011), 1–18.
- M. Crandall, P. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Functional Analysis **8** (1971), 321–340.
- P. Fitzpatrick, J. Pejsachowicz, L. Recht, *Spectral flow and bifurcation of critical points of strongly-indefinite functionals, I. General theory*, J. Funct. Anal. 162 (1999), no. 1, 52–95.
- R. Bettiol, P. Piccione, *Instability and Bifurcation*, Notices of the AMS, December 2020.

EVALUATION OF STUDENTS

In order to be evaluated, students enrolled in the course will present a one hour seminar on one of the topics discussed during the course.