# Infnite horizon optimal control problems with non-compact control space. Existence results and dynamic programming


## Abstract

In optimal control theory, infinite horizon problems may be difficult to treat when featuring large classes of admissible controls, non-concave dynamics or state constraints. On the other hand, such attributes appear frequently when the problem draws its origins from an applicative model.

The lack of compactness of the control space and/or the state space is indeed a significant penalty when trying to prove an existence result, may it concern optimal controls or the solution of some necessary conditions of optimality such as the Hamilton Jacobi Bellman equation. This is even more true if such feature goes along with a heavy non-linearity (such as non-convexity or non-concavity) in the state equation.

Dealing with a general dynamics, and with the unboundedness both of the time horizon and the control, makes *a priori* estimates on the state behaviour insufficient to use standard arguments to prove the existence of the optimal control, neither in form of the direct method nor by using more subtle tools like the approach *à la* Cesari or a suitable verification theorem. In a more subtle but not surprising way, also the theory of (viscosity) solutions to the Hamilton-Jacobi-Bellman equation "feels" the absence of compactness, as far as both existence and uniqueness are concerned.

All this is reflected in a scarcity of general results for non-compact problems in the literature.

Obviously these are key questions in the treatment of any optimal control problem, insofar as they are fundamental to the well-posedness of the theory (in the case of optimal controls) or its applicability and numerical approximation (in the case of the solutions to the Pontryagin or Hamilton-Jacobi-Bellman equations - note that the latter can be used to find optimal controls also).

Non compact and non-convex problems arise naturally in models coming from various fields of economic theory. In some cases, despite a large stream of literature, the above mentioned questions happened to remain without an answer for many years.

In this seminar we depict an original technique which we have developed in order to bypass the lack of compactness. The main application is the proof of the existence of optimal controls for three different non-compact problems resulting from two classical economic models, Ramsey-Skiba (1978) and Shallow Lake (2000). We also show how part of the tools can be used in an essential way to prove that the value function is a (bilateral) viscosity solution to the Hamilton-Jacobi-Bellman equation of the Ramsey-Skiba problem, which shows a state constraint as an additional technical difficulty.


## Outline

A natural development of this line of research is the proof of a general existence theorem for infinite horizon, non-compact multidimensional optimal control problems. The same approach could be oriented towards the study of some prominent, though not yet well understood, models of infinite dimension, such as spatial Ramsey models.

Another application of these tools could be the development of a more general theory of existence and uniqueness of viscosity solutions to HJB problems beyond the usual requirement of compactness of the domain or boundedness of the solutions.